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Equations of Motion for Charged Spinning Fluid in Bi-metric Type Theories of Gravity

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Abstract

The General theory of relativity is one of the most successful theories of gravity. Despite its successful applications, it has some difficulties in examining the behaviour of particles precisely in strong gravitational fields. Bi-metric type theories of gravity are classified as alternative theories of gravity that describing such strong gravitational fields, such as the gravitational field formed at the core of our galaxy. In order to obtain the equations of motion for spinning fluids, we use the Weyssenhoff tensor to express the spin fluid. The equations of motion for spinning fluids are derived using Euler-Lagrange equation. We present the equations of motion for spinning fluids and their corresponding spin deviation equations in some classes of Bi-metric type theories. Also, we obtain equations of motion for spinning fluids and their corresponding spin deviation for a variable mass. Moreover, we extend our study to examine the status of motion for spinning charged fluids and their corresponding spin deviation equations.

Keywords: Bi-metric type theories, Geodesic deviation, Spin density deviation, Charged spinning fluid.

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1. Introduction

Einstein's theory of gravitation (GR), has, so far, been considered to be one of the great achievements of the last century [1]. This is because of its confirmation with respect to a degree of accuracy to all gravitational observations and experiments, detected during that epoch [2]. In addition to its predictions [3], this revealed new phenomena that were detected recently in the current century, such as gravitational waves [4]. This theory has been established in the context of pseudo-Riemannian geometry depending on two principles: the general covariance and the equivalence principles. At the end of the 20th century, standard cosmology had problems due to its inability to examine the behaviour of particles in strong gravitational fields.

On one hand this prompted some authors to construct alternative theories of gravity to conceive and interpret tests of gravity [5, 6]. There are numerous attempts to formulate other theories of gravity, in this article, we devoted on one of these. One of the streams that took constructive theory of gravity alternative to GR was started by Fierz and Pauli 1939 [7]. This stream is known in the literature as Bi-metric type theories of gravity (for a review see [8]). These theories of gravity are considered promising theories to interpreting phenomena in strong field.

The importance of the bi-metric type theories of gravity stems from its ability to examine the behavior of particles in strong gravitational fields. Additionally, bi-metric type theories are considered as gauge field theories. It has numerous versions. A Class of these theories respects Lorentz invariance. While some of its versions break Lorentz invariant. Lorentz-breaking theories for massive gravity has the same progress as that of Lorentz-invariant theories, based on the experience summarized from recent developments in such theories. In what follows, we are going to display briefly some approaches of this type of theories:

(a)The Rosen Approach

In 1940, Rosen [9] proposed bi-metric theory of gravitation is, satisfying the covariance and equivalence principles. In his approach, he introduced a second metric tensor $\gamma_{\epsilon\sigma}$ corresponding to flat space, besides the metric tensor $g_{\epsilon\sigma}$. The theory's fundamental concept is that any point on the manifold is represented by two reference frames, the first of which is expressed in a flat space and the second of which is curved.

(b)The Moffat Approach

In this approach, Moffat [10] has merged the above two metrics, producing a new one defined as:

$$\hat{g}_{\epsilon\sigma} \stackrel{\text{def}}{=} g_{\epsilon\sigma} + B\partial_{\epsilon}\varphi\partial_{\sigma}\varphi, \qquad (1)$$

where, $B \; and \; \varphi$ represents a bi-scalar field. This version of bi-metric theory breaks the Lorentz invariance in the very early universe, supposing that the speed of light undergoes a first or second order phase transition in this epoch. This modification has interpreted the problem of dark energy [11], due to his proposal that speed of light is not constant in space-time or what is called a variable of the speed of light (VSL).

(c) The Milgrom Approach

According to this version it is proposed to involve two metrics as independent degrees of freedom [12], the MOND metric $g_{\epsilon\sigma}$which is responsible for the ordinary matter and an auxiliary metric $\gamma_{\epsilon\sigma}$ proposed to express twin matter.

The basic idea depends on the fact that we may create tensors from the difference between the Levi-Civita connections of the two metrics having the following form

$$C^{\alpha}_{\mu\nu} \stackrel{\text{def}}{=} \Gamma^{\alpha}_{\mu\nu} - \bar{\Gamma}^{\alpha}_{\mu\nu}. \qquad (2)$$

The above third order tensor act like gravitational accelerations of the two sectors.

The importance of BIMOND is that its ability to interpret phenomena subject to strong gravitational fields as the core of black holes [13] and describing the behaviour of galactic dark matter and dark energy. Also, it plays the role of measuring the gravitational lensing in an accurate way.

(d) The Hassan - Rosen Approach

Hossenfelder has formulated another version for bi-metric theories by proposing two different metrics one is defined on the tangent space TM while the other is in its cotangent space T^*M on a manifold M [13, 14]. Each metric has its own Levi-Cevita connection and curvature tensor. Then, two different fields are taken into account, each of which moves in accordance with a certain metric and its connection.

(e) The Verozub Approach

The bi-metric theory of gravity [15] has been extended to an alternate version by Verozub by the addition of geodesic mappings. This made it possible to describe gravity in two different geometries, one of them Riemannian space as a co-moving reference frame, and the other a Minkowski space, an inertial reference frame. Consequently, a point mass moving in a co-moving reference frame (Riemannian space) may observed to moving along a geodesic line, but in reality it is actually moving under a force field as viewed from an inertial reference frame.

In this modification, the geodesic mapping acts as gauge transformations [16]. Such a tendency makes it possible to examine the behaviour of trajectories in very strong gravitational fields such as Sgr A* [17]. As a result, this kind of description can tackle problems with strong gravity and stability problems surrounding supermassive black holes.

On the other hand, the arising notion of examining intrinsic property of matter that becomes essential in the presence of strong fields of gravity in their studies [18]. One of such staggering features is the problem of the intrinsic spin which plays a dominant role in the early stages of the universe, excluding the possibility of a cosmic singularity [19, 20].

Also, some authors have focused on studying the spinning motion because it is considered to be one of the true elements of the characteristic behaviour of objects in nature. Therefore, various attempts had been done in the domain of the theory of general relativity, beginning with Mathisson [21], and continuing with Papapetrou [22]. Mathisson-Papapetrou provides the following equations for the dynamics of spinning particles

$$\frac{DP^{\alpha}}{D\tau} = \frac{1}{2} R^{\alpha}_{\ \mu\rho\sigma} S^{\rho\sigma} U^{\mu}, \qquad (3)$$

$$\frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{D}\tau} = P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha}, \quad (4)$$

while, $P^{\alpha} = mU^{\alpha} + U_{\beta}\frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{D}\tau}$ defines the momentum of the particle, $S^{\alpha\beta}$is the spin density tensor, and U^{μ}denotes the 4-velocity of the particle. Moreover, such a type of an approach has been expanding to study spinning charged objects by Dixon [23] by means to include other non-gravitational forces such as electromagnetic. Accordingly, Equations (3) and (4) may be extended to include magnetic moments as known as the Dixon-Souriau equations in the following way has developed a method to incorporate spinning motion and (cf. [24])

$$\frac{\mathrm{D}P^{\alpha}}{\mathrm{D}\tau} = \frac{1}{2}R^{\alpha}_{\mu\rho\sigma}\, S^{\rho\sigma}U^{\mu} + q\, F^{\alpha}_{\epsilon}U^{\epsilon} + \frac{1}{2}g^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}, \quad (5)$$

$$\frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{D}\tau} = P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha} - \left(M^{\alpha\rho}F_{\rho}{}^{\beta} + M^{\beta\rho}F_{\rho}{}^{\alpha}\right), \quad (6)$$

$$\frac{\mathrm{Dq}}{\mathrm{D}\tau} = 0. \quad (7)$$

where, $M^{\epsilon\sigma}$gives electromagnetic moment and q denotes the electric charge of the particle. The problem of motion of spinning particles has been tackled in other non-Riemannian geometries such as Absolute Parallelism and Finslerian geometries [25, 26]. These works have been extended to examine the problem of spinning fluids in the context of Riemannian geometry [27].

In the present work, our main aim is to study equations of motion for spinning fluid in the context of some versions of Bi-metric theories of gravity. Accordingly, the article is arranged as follows. In Section 2 we introduce the Weyssenhoff spin tensor needed for the current application. Equations of motion for spinning fluids and their corresponding spin deviation are obtained in Section 3. In Sect. 4, we derive equations of motion for spinning fluids and their corresponding spin deviation for a variable mass. In Sect. 5, we investigate equations of motion for spinning charged fluids and their corresponding spin deviation. In Sect.6 we give some comments about the obtained equations and their forthcoming applications.

2. Weyssenhoff Spin Fluid

Through the article, we are going to derive the equations of motion for spinning fluid using the Weyssenhoff spin tensor.

On microscopic scales, the spin of the matter fields acts as a characteristic of the continuous Weyssenhoff fluid, since the Weyssenhoff fluid is a perfect fluid that has intrinsic spin. For which the spin (angular momentum) density of the matter fields can be described by the second-order skew tensor

$$S^{\alpha\beta} = -S^{\beta\alpha}. \quad (8)$$

It is a good candidate to describe particles having pole-dipole moments. To describe the motion of a spinning fluid the second-order tensor $S^{\alpha\beta}$ must be extended to another one that can describe multi-pole moments for extended objects. Accordingly, we use the Weyssenhoff tensor for spinning fluid which is postulated to be (cf. [28])

$$S^{\sigma\alpha\beta} = S^{\alpha\beta}U^{\sigma}, \quad (9)$$

where, $S^{\sigma\alpha\beta}$ is a third-order skew-tensor in the last two indices which is chosen to represent the spin density of the fluid, $U^{\sigma} = \frac{dx^{\sigma}}{d\tau}$ is the unit tangent vector and τ is a parameter varying along the curve. In the case of the spinning fluid motion having a precession the Weyssenhoff tensor can be written as

$$S^{\sigma\alpha\beta} = S^{\alpha\beta} P^{\sigma}, \tag{10}$$

where, P^{σ} is the momentum of the particle and is defined as:

$$P^{\sigma} \overset{\text{def}}{=} mU^{\sigma} + U_{\epsilon} \frac{\mathrm{D}S^{\epsilon\sigma}}{\mathrm{D}\tau}. \tag{11}$$

While for variable mass, the Weyssenhoff tensor is expressed as

$$\check{S}^{\sigma\alpha\beta} = m(\tau) S^{\alpha\beta} U^{\sigma}, \tag{12}$$

where, $m(\tau)$ is a function of the parameter τ representing variable mass.

3. Equations of Motion for Spinning Fluids and its Corresponding Spin Deviation

In this section, we will derive the equations of motion for spinning fluids and their corresponding spin deviation in the bi-metric type theories. However, it is worth to mention that the equation of spinning motion for spinning particle for these versions of Bi-metric type theories have been obtained in [29].

3.1. Rosen Approach

Equations of motion in the context of Rosen's approach in the case of $P^{\epsilon} = mU^{\epsilon}$ can be derived using the following Lagrangian function [29]

$$L = (g_{\epsilon\sigma} - \gamma_{\epsilon\sigma}) U^{\epsilon} \frac{\underline{\nabla}\Psi^{\sigma}}{\underline{\nabla}\tau} + S_{\epsilon\sigma} \frac{\underline{\nabla}\Psi^{\epsilon\sigma}}{\underline{\nabla}\tau} + \frac{1}{2} R_{\mu\nu\epsilon\sigma} S^{\epsilon\sigma} U^{\nu} \Psi^{\mu}. \tag{13}$$

where, $g_{\epsilon\sigma}$ and $\gamma_{\epsilon\sigma}$ are the metric tensors of the curved space and the flat space respectively, and $R_{\mu\nu\epsilon\sigma}$ is the curvature tensor formed from the metric tensor $g_{\epsilon\sigma}$. While, $\gamma_{\epsilon\sigma}$ having a vanishing curvature tensor. And $\frac{\underline{\nabla}}{\underline{\nabla}\tau}$, is an operator that characterizes covariant derivative, such that for an arbitrary vector A^{α} the covariant derivative is defined as:

$$\frac{\underline{\nabla}A^{\alpha}}{\underline{\nabla}\tau} = \frac{\mathrm{d}A^{\alpha}}{\mathrm{d}\tau} + \Delta^{\alpha}_{\mu\nu} A^{\mu} U^{\nu}, \tag{14}$$

as, $\Delta^{\alpha}_{\mu\nu} \overset{\text{def}}{=} \Gamma^{\alpha}_{\mu\nu} - \underline{\Gamma}^{\alpha}_{\mu\nu}$ where $\Gamma^{\alpha}_{\mu\nu}$ and $\underline{\Gamma}^{\alpha}_{\mu\nu}$ are the affine connection of the curved and flat space, respectively. To obtain the path equations one have to apply the following Euler-Lagrange equation with respect to deviation vector Ψ^{α} and to the spin deviation tensor $\Psi^{\alpha\beta}$ which have the forms:

$$\frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^{\alpha}} - \frac{\partial L}{\partial \Psi^{\alpha}} = 0, \tag{15}$$

and,

$$\frac{d}{dS} \frac{\partial L}{\partial \dot{\Psi}^{\alpha\beta}} - \frac{\partial L}{\partial \Psi^{\alpha\beta}} = 0. \tag{16}$$

One gets,

$$\frac{\underline{\nabla}U^{\alpha}}{\underline{\nabla}\tau} = \frac{1}{2} R^{\alpha}_{\ \mu\rho\sigma} S^{\rho\sigma} U^{\mu}, \tag{17}$$

and,

$$\frac{\underline{\nabla} S^{\alpha\beta}}{\underline{\nabla}\tau} = 0. \tag{18}$$

Consequently, using the Weyssenhoff tensor (9), the equation of spinning motion for the spinning fluid will have the form

$$\frac{\underline{\nabla} S^{\alpha\beta\gamma}}{\underline{\nabla}\tau} = \frac{1}{2} R^{\alpha}{}_{\mu\epsilon\sigma} S^{\epsilon\sigma} S^{\beta\gamma} U^{\mu}. \tag{19}$$

To obtain the equation of spinning density deviation, one has to apply the condition [30]

$$S^{\alpha\beta\gamma}{}_{;\delta}\Psi^{\delta} = \Psi^{\alpha\beta\gamma}{}_{;\delta}U^{\delta}, \tag{20}$$

which implies using the following commutation relation

$$\left(S^{\alpha\beta\gamma}{}_{;\delta\rho} - S^{\alpha\beta\gamma}{}_{;\rho\delta}\right)\Psi^{\delta}U^{\rho} = S^{\epsilon[\beta\gamma} R^{\alpha]}{}_{\epsilon\delta\rho} U^{\delta} \Psi^{\rho}. \tag{21}$$

Accordingly; we can obtain the equation of spinning density deviation to have the form:

$$\frac{\underline{\nabla}^2 \Psi^{\alpha\beta\gamma}}{\underline{\nabla}\tau^2} = S^{\rho[\beta\gamma} R^{\alpha]}{}_{\rho\epsilon\sigma} U^{\epsilon} \Psi^{\sigma} + \frac{1}{2}\Big[\left(R^{\alpha}{}_{\mu\epsilon\sigma} U^{\mu} S^{\epsilon\sigma} S^{\beta\gamma}\right)_{;\delta} + \left(R^{\alpha}{}_{\mu\epsilon\sigma} U^{\mu} S^{\epsilon\sigma} S^{\beta\gamma}\right)_{|\delta}\Big]\Psi^{\delta}, \tag{22}$$

where, the semicolon (;) and stroke (|) are used as infix operators characterize tensor derivatives using connections of curved and flat spaces, respectively.

While in the case of motion with precession, i.e. $P^{\epsilon} = mU^{\epsilon} + U_{\sigma}\frac{\underline{\nabla} S^{\epsilon\sigma}}{\underline{\nabla}\tau}$, the Lagrangian of spinning motion written as

$$L = (g_{\epsilon\sigma} - \gamma_{\epsilon\sigma})P^{\epsilon}\frac{\underline{\nabla}\Psi^{\sigma}}{\underline{\nabla}\tau} + S_{\epsilon\sigma}\frac{\underline{\nabla}\Psi^{\epsilon\sigma}}{\underline{\nabla}\tau} + \frac{1}{2}R_{\mu\nu\epsilon\sigma} S^{\epsilon\sigma} U^{\mu}\Psi^{\nu} + 2P_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}. \tag{23}$$

By applying the variation to the Lagrangian (23) with respect to Ψ^{σ} and $\Psi^{\epsilon\sigma}$, one obtains

$$\frac{\underline{\nabla} P^{\alpha}}{\underline{\nabla}\tau} = \frac{1}{2} R^{\alpha}{}_{\mu\rho\sigma} S^{\rho\sigma} U^{\mu}, \tag{24}$$

and,

$$\frac{\underline{\nabla} S^{\alpha\beta}}{\underline{\nabla}\tau} = 2P^{[\alpha}U^{\beta]}. \tag{25}$$

Using equations (24) and (25), then the equation of spinning motion can be written as

$$\frac{\underline{\nabla} S^{\alpha\beta\gamma}}{\underline{\nabla}\tau} = 2P^{\alpha}P^{[\beta}U^{\gamma]} + \frac{1}{2} R^{\alpha}{}_{\mu\epsilon\sigma} S^{\epsilon\sigma} S^{\beta\gamma} U^{\mu}. \tag{26}$$

Applying (21) and the condition (20), then the equation of spinning density deviation becomes

$$\frac{\underline{\nabla}^2 \Psi^{\alpha\beta\gamma}}{\underline{\nabla}\tau^2} = S^{\rho[\beta\gamma} R^{\alpha]}{}_{\rho\epsilon\sigma} U^{\epsilon} \Psi^{\sigma} + 2\Big[\left(P^{\alpha}P^{[\beta}U^{\gamma]}\right)_{;\delta} + \left(P^{\alpha}P^{[\beta}U^{\gamma]}\right)_{|\delta}\Big]\Psi^{\delta} + \frac{1}{2}\Big[\left(R^{\alpha}{}_{\mu\epsilon\sigma} U^{\mu} S^{\epsilon\sigma} S^{\beta\gamma}\right)_{;\delta} + \left(R^{\alpha}{}_{\mu\epsilon\sigma} U^{\mu} S^{\epsilon\sigma} S^{\beta\gamma}\right)_{|\delta}\Big]\Psi^{\delta} \tag{27}$$

3.2. Moffat's Approach

The Lagrangian representing spinning motion according to Moffat's approach in case of $P^{\epsilon} = mU^{\epsilon}$, have the form:

$$L = \hat{g}_{\epsilon\sigma} U^{\epsilon} \frac{\hat{\nabla}\Psi^{\sigma}}{\hat{\nabla}\tau} + S_{\epsilon\sigma} \frac{\hat{\nabla}\Psi^{\epsilon\sigma}}{\hat{\nabla}\tau} + \frac{1}{2}\hat{L}_{\mu\nu\epsilon\sigma} S^{\epsilon\sigma} U^{\nu} \Psi^{\mu}. \tag{28}$$

where,

$$\hat{L}^{\alpha}{}_{\mu\nu\sigma} \stackrel{\text{def}}{=} \hat{\Gamma}^{\alpha}{}_{\mu\sigma,\nu} - \hat{\Gamma}^{\alpha}{}_{\mu\nu,\sigma} + \hat{\Gamma}^{\epsilon}{}_{\mu\sigma}\hat{\Gamma}^{\alpha}{}_{\epsilon\nu} - \hat{\Gamma}^{\epsilon}{}_{\mu\nu}\hat{\Gamma}^{\alpha}{}_{\epsilon\sigma},$$

defines the curvature tensor, and the operator $\frac{\hat{\nabla}}{\hat{\nabla}\tau}$ defines the covariant derivative w.r.to the parameter τ such that for any arbitrary vector A^{α}, we have

$$\frac{\hat{\nabla}A^{\alpha}}{\hat{\nabla}\tau} = \frac{\mathrm{d}A^{\alpha}}{\mathrm{d}\tau} + \hat{\Gamma}^{\alpha}_{\mu\nu} A^{\mu} U^{\nu},$$

By operating the Euler-Largrange equations (15) and (16) to (28), one obtains

$$\frac{\hat{\nabla}U^{\alpha}}{\hat{\nabla}\tau} = \frac{1}{2}\hat{L}^{\alpha}{}_{\mu\rho\sigma} S^{\rho\sigma} U^{\mu}. \tag{29}$$

and

$$\frac{\hat{\nabla}S^{\alpha\beta}}{\hat{\nabla}\tau} = 0. \tag{30}$$

Meanwhile, from Weyssenhoff tensor (9), the equation of motion for spinning fluid can be written as

$$\frac{\hat{\nabla}S^{\alpha\beta\gamma}}{\hat{\nabla}\tau} = \frac{1}{2}\hat{L}^{\alpha}{}_{\mu\epsilon\sigma} S^{\epsilon\sigma} S^{\beta\gamma} U^{\mu}. \tag{31}$$

In the context of Moffat's approach the condition (20), can be rewritten as:

$$S^{\alpha\beta\gamma}{}_{||\delta}\Psi^{\delta} = \Psi^{\alpha\beta\gamma}{}_{||\delta}\mathrm{U}^{\delta}, \tag{32}$$

and the commutation relation will have the form

$$\left(S^{\alpha\beta\gamma}{}_{||\delta\rho} - S^{\alpha\beta\gamma}{}_{||\rho\delta}\right)\Psi^{\delta}\mathrm{U}^{\rho} = S^{\epsilon[\beta\gamma}\hat{L}^{\alpha]}{}_{\epsilon\delta\rho}\, U^{\delta}\, \Psi^{\rho}. \tag{33}$$

Accordingly, by using the relation (33) and the condition (32), we can obtain the equation of spin density deviation as

$$\frac{\hat{\nabla}^2\Psi^{\alpha\beta\gamma}}{\hat{\nabla}\tau^2} = S^{\rho[\beta\gamma}\hat{L}^{\alpha]}{}_{\rho\epsilon\sigma}\, U^{\epsilon}\, \Psi^{\sigma} + \frac{1}{2}\left(\hat{L}^{\alpha}{}_{\mu\epsilon\sigma}\, U^{\mu}\, S^{\epsilon\sigma}\, S^{\beta\gamma}\right)_{||\delta}\Psi^{\delta}. \tag{34}$$

where, the double-stroke (||) is an infix operator used to characterize tensor derivatives using connection $\hat{\Gamma}^{\alpha}_{\mu\nu}$.

While in case of $P^{\epsilon} = mU^{\epsilon} + U_{\sigma}\frac{\hat{\nabla}S^{\epsilon\sigma}}{\hat{\nabla}\tau}$, the Lagrangian of spinning can be written as

$$L = \hat{g}_{\epsilon\sigma} P^{\epsilon} \frac{\hat{\nabla}\Psi^{\sigma}}{\hat{\nabla}\tau} + S_{\epsilon\sigma} \frac{\hat{\nabla}\Psi^{\epsilon\sigma}}{\hat{\nabla}\tau} + \frac{1}{2}\hat{L}_{\mu\nu\epsilon\sigma} S^{\epsilon\sigma} U^{\nu} \Psi^{\mu} + 2P_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}. \tag{35}$$

By operating the variation for the Lagrangian (35) with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, one gets

$$\frac{\hat{\nabla} P^{\alpha}}{\hat{\nabla}\tau} = \frac{1}{2}\hat{L}^{\alpha}{}_{\mu\epsilon\sigma}\, S^{\epsilon\sigma} U^{\mu}, \tag{36}$$

and

$$\frac{\hat{\nabla} S^{\alpha\beta}}{\hat{\nabla}\tau} = 2P^{[\alpha} U^{\beta]}. \tag{37}$$

Following the same technique mentioned above we can derive the equation of motion and the spin density deviation equation as follow

$$\frac{\hat{\nabla} S^{\alpha\beta\gamma}}{\hat{\nabla}\tau} = 2P^{\alpha} P^{[\beta} U^{\gamma]} + \frac{1}{2}\hat{L}^{\alpha}{}_{\mu\epsilon\sigma} S^{\epsilon\sigma}\, S^{\beta\gamma} U^{\mu}, \tag{38}$$

and

$$\frac{\hat{\nabla}^{2} \Psi^{\alpha\beta\gamma}}{\hat{\nabla}\tau^{2}} = S^{\rho[\beta\gamma}\hat{L}^{\alpha]}{}_{\rho\epsilon\sigma}\, U^{\epsilon}\, \Psi^{\sigma} + \left(2P^{\alpha} P^{[\beta} U^{\gamma]} + \frac{1}{2}\hat{L}^{\alpha}{}_{\mu\epsilon\sigma}\, U^{\mu}\, S^{\epsilon\sigma}\, S^{\beta\gamma}\right)_{||\delta} \Psi^{\delta}. \tag{39}$$

3.3. BIMOND Type Theories

The Lagrangian function for spinning motion in case of $P^{\epsilon} = mU^{\epsilon}$ in the context of BIMOND type theories, can be written as

$$L = \bar{g}_{\epsilon\sigma} U^{\epsilon} \frac{\bar{\nabla}\Psi^{\sigma}}{\bar{\nabla}\tau} + S_{\epsilon\sigma}\, \frac{\bar{\nabla}\Psi^{\epsilon\sigma}}{\bar{\nabla}\tau} + \frac{1}{2}\left(R_{\mu\nu\epsilon\sigma} - \bar{N}_{\mu\nu\epsilon\sigma}\right) S^{\epsilon\sigma} U^{\nu} \Psi^{\mu}, \tag{40}$$

where,

$$\bar{N}^{\alpha}_{\mu\nu\sigma} \stackrel{\text{def}}{=} \bar{\Gamma}^{\alpha}{}_{\mu\sigma,\nu} - \bar{\Gamma}^{\alpha}{}_{\mu\nu,\sigma} + \bar{\Gamma}^{\epsilon}{}_{\mu\sigma}\bar{\Gamma}^{\alpha}{}_{\epsilon\nu} - \bar{\Gamma}^{\epsilon}{}_{\mu\nu}\bar{\Gamma}^{\alpha}{}_{\epsilon\sigma}.$$

By taking the variation with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, to the Lagrangian (40), then one can obtain the path equations as:

$$\frac{\bar{\nabla} U^{\alpha}}{\bar{\nabla}\tau} = \frac{1}{2}\left(R^{\alpha}_{\mu\rho\sigma} - \bar{N}^{\alpha}_{\mu\rho\sigma}\right) S^{\rho\sigma} U^{\mu}. \tag{41}$$

and

$$\frac{\bar{\nabla} S^{\alpha\beta}}{\bar{\nabla}\tau} = 0. \tag{42}$$

Accordingly, we can obtain the equation of motion for spinning fluid using the Weyssenhoff tensor (9) in the form:

$$\frac{\bar{\nabla} S^{\alpha\beta\gamma}}{\bar{\nabla}\tau} = \frac{1}{2}\left(R^{\alpha}_{\mu\epsilon\sigma} - \bar{N}^{\alpha}_{\mu\epsilon\sigma}\right) S^{\epsilon\sigma}\, S^{\beta\gamma} U^{\mu}. \tag{43}$$

Using the condition (20), and

$$S^{\alpha\beta\gamma}{}_{|_{+}^{\delta}}\Psi^{\delta} = \Psi^{\alpha\beta\gamma}{}_{|_{+}^{\delta}}U^{\delta}, \qquad (44)$$

together with the commutation relation (21) and the following one

$$\left(S^{\alpha\beta\gamma}{}_{|_{++}^{\delta\rho}} - S^{\alpha\beta\gamma}{}_{|_{++}^{\rho\delta}}\right)\Psi^{\delta}U^{\rho} = S^{\epsilon[\beta\gamma}\bar{N}^{\alpha]}{}_{\epsilon\delta\rho}\, U^{\delta}\,\Psi^{\rho}. \qquad (45)$$

We can derive the spin density deviation tensor to become

$$\frac{\bar{\nabla}^2\Psi^{\alpha\beta\gamma}}{\bar{\nabla}\tau^2} = \left(S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma} - S^{\rho[\beta\gamma}\bar{N}^{\alpha]}{}_{\rho\epsilon\sigma}\right)\, U^{\epsilon}\,\Psi^{\sigma} + \frac{1}{2}\left(R^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}\,S^{\beta\gamma}\right)_{;\delta}\Psi^{\delta} + \frac{1}{2}\left(\bar{N}^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}\,S^{\beta\gamma}\right)_{|_{+}^{\delta}}\Psi^{\delta}. \qquad (46)$$

While, the double-stroke (|) together with the (+) sign are infix operators used to represent tensor derivatives using connection$\bar{\Gamma}^{\alpha}_{\mu\nu}$.

But, in case of motion with precession, i.e.$\bar{P}^{\epsilon} = mU^{\epsilon} + U_{\sigma}\frac{\bar{\nabla}S^{\epsilon\sigma}}{\bar{\nabla}\tau}$, the Lagrangian will have the form:

$$L = \bar{g}_{\epsilon\sigma}U^{\epsilon}\frac{\bar{\nabla}\Psi^{\sigma}}{\bar{\nabla}\tau} + S_{\epsilon\sigma}\,\frac{\bar{\nabla}\Psi^{\epsilon\sigma}}{\bar{\nabla}\tau} + \frac{1}{2}\left(R_{\mu\nu\epsilon\sigma} - \bar{N}_{\mu\nu\epsilon\sigma}\right)\, S^{\epsilon\sigma}U^{\nu}\Psi^{\mu} + 2\bar{P}_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}. \qquad (47)$$

Taking variation to (47) with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, the path equations can be written as:

$$\frac{\bar{\nabla}P^{\alpha}}{\bar{\nabla}\tau} = \frac{1}{2}\left(R^{\alpha}_{\ \mu\epsilon\sigma} - \bar{N}^{\alpha}_{\ \mu\epsilon\sigma}\right)\, S^{\epsilon\sigma}U^{\mu}. \qquad (48)$$

and,

$$\frac{\bar{\nabla}S^{\alpha\beta}}{\bar{\nabla}\tau} = 2P^{[\alpha}U^{\beta]}. \qquad (49)$$

Using the Weyssenhoff tensor (10), then we obtain the equation of motion for spinning fluid as follow:

$$\frac{\bar{\nabla}S^{\alpha\beta\gamma}}{\bar{\nabla}\tau} = 2P^{\alpha}P^{[\beta}U^{\gamma]} + \frac{1}{2}\left(R^{\alpha}_{\ \mu\epsilon\sigma} - \bar{N}^{\alpha}_{\ \mu\epsilon\sigma}\right)\, S^{\epsilon\sigma}\,S^{\beta\gamma}U^{\mu}. \qquad (50)$$

Using relations (21) and (45) together with conditions (20) and (44), we can obtain the equation of spin density deviation as

$$\frac{\bar{\nabla}^2\Psi^{\alpha\beta\gamma}}{\bar{\nabla}\tau^2} = \left(S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma} - S^{\rho[\beta\gamma}\bar{N}^{\alpha]}{}_{\rho\epsilon\sigma}\right)U^{\epsilon}\,\Psi^{\sigma} + 2\left[\left(\bar{P}^{\alpha}\bar{P}^{[\beta}U^{\gamma]}\right)_{;\delta} + \left(\bar{P}^{\alpha}\bar{P}^{[\beta}U^{\gamma]}\right)_{|_{+}^{\delta}}\right]\Psi^{\delta} + \frac{1}{2}\left[\left(R^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}\,S^{\beta\gamma}\right)_{;\delta} + \left(\bar{N}^{\alpha}_{\ \mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}\,S^{\beta\gamma}\right)_{|_{+}^{\delta}}\right]\Psi^{\delta}. \qquad (51)$$

3.4. Hassan-Rosen Approach: Bi-gravity type theories

According to this version, the two suggested metrics $g_{\epsilon\sigma}$and $h_{\epsilon\sigma}$ are chosen to define two distinct field equations one describe the matter while the second for the twin matter.

The corresponding Lagrangian of spinning motion in case of $P^{\epsilon} = mU^{\epsilon}$ and $\tilde{P}^{\epsilon} = \tilde{m}U^{\epsilon}$ can be expressed as:

$$L = g_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma}{}_{;\alpha}U^{\alpha} + h_{\epsilon\sigma}V^{\epsilon}\Phi^{\sigma}{}_{;\alpha}V^{\alpha} + S_{\epsilon\sigma}\,\Psi^{\epsilon\sigma}{}_{;\alpha}U^{\alpha} + \tilde{S}_{\epsilon\sigma}\,\Phi^{\epsilon\sigma}{}_{;\alpha}V^{\alpha}$$

$$+\frac{1}{2}R_{\mu\nu\epsilon\sigma}S^{\epsilon\sigma}U^{\nu}\Psi^{\mu} + \frac{1}{2}\tilde{R}_{\mu\nu\epsilon\sigma}\tilde{S}^{\epsilon\sigma}V^{\nu}\Phi^{\mu}\,. \quad (52)$$

WhereV^{α}, $\Phi^{\sigma}$and $\tilde{S}^{\epsilon\sigma}$ characterizing the twin unit tangent vector, the twin deviation vector and the twin spinning tensor, respectively. And $\tilde{R}_{\mu\nu\epsilon\sigma}$ defines the curvature tensor formed by the metric$h_{\epsilon\sigma}$.

Accordingly, the path equations can be obtained by applying variation with respect to the deviation vector Φ^{α} and Ψ^{α}, to the Lagrangian (52), one gets:

$$\frac{\mathrm{D}U^{\alpha}}{\mathrm{D}\tau} = \frac{1}{2}R^{\alpha}{}_{\mu\rho\sigma}\,S^{\rho\sigma}U^{\mu}. \quad (53)$$

$$\frac{\mathrm{D}V^{\alpha}}{\mathrm{D}\lambda} = \frac{1}{2}\tilde{R}^{\alpha}{}_{\mu\rho\sigma}\,\tilde{S}^{\rho\sigma}V^{\mu}. \quad (54)$$

While the variation with respect to the deviation tensor $\Phi^{\alpha\beta}$ & $\Psi^{\alpha\beta}$, gives rise to the following equations

$$\frac{\mathrm{D}S^{\alpha\beta}}{\mathrm{D}\tau} = 0. \quad (55)$$

And,

$$\frac{\mathrm{D}\tilde{S}^{\alpha\beta}}{\mathrm{D}\lambda} = 0. \quad (56)$$

From the Weyssenhoff tensor, by using equations (53-56) then we can obtain the equations of motion for spinning fluid, to become

$$\frac{\mathrm{D}S^{\alpha\beta\gamma}}{\mathrm{D}\tau} = \frac{1}{2}R^{\alpha}{}_{\mu\epsilon\sigma}S^{\epsilon\sigma}\,S^{\beta\gamma}U^{\mu}. \quad (57)$$

$$\frac{\mathrm{D}\tilde{S}^{\alpha\beta\gamma}}{\mathrm{D}\lambda} = \frac{1}{2}\tilde{R}^{\alpha}{}_{\mu\epsilon\sigma}\tilde{S}^{\epsilon\sigma}\,\tilde{S}^{\beta\gamma}V^{\mu}. \quad (58)$$

Consequently, by using the relation (21) and the condition (20), we can obtain the equation of spin density deviation as

$$\frac{\mathrm{D}^2\Psi^{\alpha\beta\gamma}}{\mathrm{D}\tau^2} = S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma}\,U^{\epsilon}\,\Psi^{\sigma} + \frac{1}{2}\left(R^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}\,S^{\beta\gamma}\right)_{;\delta}\Psi^{\delta}$$

Using the condition

$$S^{\alpha\beta\gamma}{}_{|||\delta}\Phi^{\delta} = \Phi^{\alpha\beta\gamma}{}_{|||\delta}\mathrm{U}^{\delta}, \quad (59)$$

and following commutation relation

$$\left(S^{\alpha\beta\gamma}{}_{|||\delta\rho} - S^{\alpha\beta\gamma}{}_{|||\rho\delta}\right)\Phi^{\delta}\mathrm{U}^{\rho} = S^{\epsilon[\beta\gamma}\tilde{R}^{\alpha]}{}_{\epsilon\delta\rho}\,U^{\delta}\,\Phi^{\rho}, \quad (60)$$

we can obtain the spin density deviation tensor to become

$$\frac{D^2\Phi^{\alpha\beta\gamma}}{D\lambda^2} = \tilde{S}^{\rho[\beta\gamma}\tilde{R}^{\alpha]}{}_{\rho\epsilon\sigma}\,V^{\epsilon}\,\Phi^{\sigma} + \frac{1}{2}\left(\tilde{R}^{\alpha}{}_{\mu\epsilon\sigma}\,V^{\mu}\,\tilde{S}^{\epsilon\sigma}\,\tilde{S}^{\beta\gamma}\right)_{|||\delta}\Phi^{\delta}. \qquad (61)$$

In case of $P^{\epsilon} = mU^{\epsilon} + U_{\sigma}\frac{DS^{\epsilon\sigma}}{D\tau}$ and $\tilde{P}^{\epsilon} = \tilde{m}V^{\epsilon} + V_{\sigma}\frac{D\tilde{S}^{\epsilon\sigma}}{D\lambda}$, the Lagrangian function can be written as

$$L = g_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma}{}_{;\alpha}U^{\alpha} + h_{\epsilon\sigma}V^{\epsilon}\Phi^{\sigma}{}_{|||\alpha}V^{\alpha} + S_{\epsilon\sigma}\,\Psi^{\epsilon\sigma}{}_{;\alpha}U^{\alpha} + \tilde{S}_{\epsilon\sigma}\,\Phi^{\epsilon\sigma}{}_{|||\alpha}V^{\alpha} + 2P_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}$$

$$+\frac{1}{2}R_{\mu\nu\epsilon\sigma}S^{\epsilon\sigma}U^{\nu}\Psi^{\mu} + \frac{1}{2}\tilde{R}_{\mu\nu\epsilon\sigma}\tilde{S}^{\epsilon\sigma}V^{\nu}\Phi^{\mu} + 2\tilde{P}_{[\epsilon}V_{\sigma]}\Phi^{\epsilon\sigma}. \qquad (62)$$

By operating the variation to the Lagrangian (62) with respect to Φ^{α}, Ψ^{α} ,$\Phi^{\alpha\beta}$and $\Psi^{\alpha\beta}$, one can obtain the following set of equation

$$\left.\begin{aligned} \frac{DP^{\alpha}}{D\tau} &= \frac{1}{2}R^{\alpha}{}_{\mu\rho\sigma}\,S^{\rho\sigma}U^{\mu}. \\ \frac{D\tilde{P}^{\alpha}}{D\lambda} &= \frac{1}{2}\tilde{R}^{\alpha}{}_{\mu\rho\sigma}\,\tilde{S}^{\rho\sigma}V^{\mu}. \\ \frac{DS^{\alpha\beta}}{D\tau} &= 2P^{[\alpha}U^{\beta]}. \\ \frac{D\tilde{S}^{\alpha\beta}}{D\lambda} &= 2\tilde{P}^{[\alpha}V^{\beta]}. \end{aligned}\right\} \qquad (63)$$

We obtain the equation of spinning motion for spinning motion, to have the form

$$\frac{DS^{\alpha\beta\gamma}}{D\tau} = 2P^{\alpha}P^{[\beta}U^{\gamma]} + \frac{1}{2}\,R^{\alpha}{}_{\mu\epsilon\sigma}\,S^{\epsilon\sigma}\,S^{\beta\gamma}U^{\mu}. \qquad (64)$$

$$\frac{D\tilde{S}^{\alpha\beta\gamma}}{D\lambda} = 2\tilde{P}^{\alpha}\tilde{P}^{[\beta}V^{\gamma]} + \frac{1}{2}\,\tilde{R}^{\alpha}{}_{\mu\epsilon\sigma}\,\tilde{S}^{\epsilon\sigma}\,\tilde{S}^{\beta\gamma}V^{\mu}. \qquad (65)$$

Following the same technique mentioned in the previous section to derive the equation of spin density deviation, we get

$$\frac{D^2\Psi^{\alpha\beta\gamma}}{D\tau^2} = S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma}U^{\epsilon}\,\Psi^{\sigma} + \left(2P^{\alpha}P^{[\beta}U^{\gamma]} + \frac{1}{2}R^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}\,S^{\beta\gamma}\right)_{;\delta}\Psi^{\delta}, (66)$$

$$\frac{D^2\Phi^{\alpha\beta\gamma}}{D\lambda^2} = S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma}U^{\epsilon}\,\Phi^{\sigma} + \left(2\tilde{P}^{\alpha}\tilde{P}^{[\beta}V^{\gamma]} + \frac{1}{2}\tilde{R}^{\alpha}{}_{\mu\epsilon\sigma}\,V^{\mu}\,\tilde{S}^{\epsilon\sigma}\,\tilde{S}^{\beta\gamma}\right)_{|||\delta}\Phi^{\delta}. (67)$$

3.5. The Verozub Approach: Bi-metric invariant-gravitation theory

In case of $P^{\epsilon} = mU^{\epsilon}$, the Lagrangian of spinning motion will have the following form:

$$L = \Upsilon_{\epsilon\sigma}(\psi) U^{\epsilon} \frac{\underline{\underline{D}}\Psi^{\sigma}}{\underline{D}\tau} + S_{\epsilon\sigma} \frac{\underline{\underline{D}}\Psi^{\epsilon\sigma}}{\underline{D}\tau} + \frac{1}{2} K_{\mu\nu\epsilon\sigma} S^{\epsilon\sigma} U^{\nu} \Psi^{\mu}. \quad (68)$$

where, $\Upsilon_{\epsilon\sigma}(\psi)$ is the combined metric under the effect of a given field ψ, and its corresponding curvature is defined as

$$K^{\alpha}_{\ \mu\nu\sigma} \stackrel{\text{def}}{=} \bar{\bar{\Gamma}}^{\alpha}_{\ \mu\sigma,\nu} - \bar{\bar{\Gamma}}^{\alpha}_{\ \mu\nu,\sigma} + \bar{\bar{\Gamma}}^{\epsilon}_{\ \mu\sigma} \bar{\bar{\Gamma}}^{\alpha}_{\ \epsilon\nu} - \bar{\bar{\Gamma}}^{\epsilon}_{\ \mu\nu} \bar{\bar{\Gamma}}^{\alpha}_{\ \epsilon\sigma}.$$

Similarly, the path equations can be obtained by operating the variation with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, to the Lagrangian (71) as follow:

$$\frac{\underline{\underline{D}}U^{\alpha}}{\underline{D}\tau} = \frac{1}{2} K_{\mu\nu\epsilon\sigma} S^{\rho\sigma} U^{\mu}, \quad (69)$$

and,

$$\frac{\underline{\underline{D}}S^{\alpha\beta}}{\underline{D}\tau} = 0. \quad (70)$$

Accordingly, by using the equations (69) and (70) and Weyssenhoff tensor (9), we get:

$$\frac{\underline{\underline{D}}S^{\alpha\beta\gamma}}{\underline{D}\tau} = \frac{1}{2} K^{\alpha}_{\ \mu\epsilon\sigma} S^{\epsilon\sigma} S^{\beta\gamma} U^{\mu}. \quad (71)$$

From the condition (20), which can be rewritten as

$$S^{\alpha\beta\gamma}{}_{||^{\delta}_{+}} \Psi^{\delta} = \Psi^{\alpha\beta\gamma}{}_{||^{\delta}_{+}} U^{\delta}, \quad (72)$$

together with the following relation

$$\left(S^{\alpha\beta\gamma}{}_{||^{\delta\rho}_{++}} - S^{\alpha\beta\gamma}{}_{||^{\rho\delta}_{++}} \right) \Psi^{\delta} U^{\rho} = S^{\epsilon[\beta\gamma} K^{\alpha]}_{\ \epsilon\delta\rho} U^{\delta} \Psi^{\rho}. \quad (73)$$

We get,

$$\frac{\underline{\underline{D}}^2\Psi^{\alpha\beta\gamma}}{\underline{D}\tau^2} = S^{\rho[\beta\gamma} K^{\alpha]}_{\ \rho\epsilon\sigma} U^{\epsilon} \Psi^{\sigma} + \frac{1}{2} \left(K^{\alpha}_{\ \mu\epsilon\sigma} U^{\mu} S^{\epsilon\sigma} S^{\beta\gamma} \right)_{||^{\delta}_{+}} \Psi^{\delta}. \quad (74)$$

Moreover, the Lagrangian of spinning motion in case of $\check{P}^{\epsilon} = mU^{\epsilon} + U_{\sigma} \frac{\bar{\nabla} S^{\epsilon\sigma}}{\bar{\nabla}\tau}$ can be expressed as:

$$L = \Upsilon_{\epsilon\sigma}(\psi) U^{\epsilon} \frac{\underline{\underline{D}}\Psi^{\sigma}}{\underline{D}\tau} + S_{\epsilon\sigma} \frac{\underline{\underline{D}}\Psi^{\epsilon\sigma}}{\underline{D}\tau} + \frac{1}{2} K_{\mu\nu\epsilon\sigma} S^{\epsilon\sigma} U^{\nu} \Psi^{\mu} + 2\check{P}_{[\epsilon} U_{\sigma]} \Psi^{\epsilon\sigma}. \quad (75)$$

Accordingly, by taking the variation with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, one obtains

$$\frac{\underline{\mathrm{D}}\check{P}^{\alpha}}{\underline{\mathrm{D}}\tau} = \frac{1}{2} K^{\alpha}_{\ \mu\epsilon\sigma}\, S^{\epsilon\sigma} U^{\mu}, \tag{76}$$

and,

$$\frac{\underline{\mathrm{D}}S^{\alpha\beta}}{\underline{\mathrm{D}}\tau} = 2\check{P}^{[\alpha} U^{\beta]}. \tag{77}$$

Following the same methods mentioned above we can derive the equation of motion and the spin density deviation equation as follow

$$\frac{\underline{\mathrm{D}}S^{\alpha\beta\gamma}}{\underline{\mathrm{D}}\tau} = 2\check{P}^{\alpha}\check{P}^{[\beta} U^{\gamma]} + \frac{1}{2} K^{\alpha}_{\ \mu\epsilon\sigma}\, S^{\epsilon\sigma}\, S^{\beta\gamma} U^{\mu}. \tag{78}$$

$$\frac{\underline{\mathrm{D}}^2\Psi^{\alpha\beta\gamma}}{\underline{\mathrm{D}}\tau^2} = S^{\rho[\beta\gamma} K^{\alpha]}_{\ \rho\epsilon\sigma} U^{\epsilon}\, \Psi^{\sigma} + \left[2\check{P}^{\alpha}\check{P}^{[\beta} U^{\gamma]} + \frac{1}{2} K^{\alpha}_{\ \mu\epsilon\sigma}\, U^{\mu}\, S^{\epsilon\sigma}\, S^{\beta\gamma}\right]_{\underset{+}{||}\delta} \Psi^{\delta}. \tag{79}$$

4. Equations of Motion for Spinning Fluids and its corresponding spin deviation: Variable Mass

In this section, we are going to derive spinning and spinning deviation equation in case of motion without precession recalling that in case of $P^{\epsilon} \neq mU^{\epsilon}$ implies that mass is constant [31].

4.1. Rosen's Approach

We suggest the Lagrangian in the context of Rosen's approach in case of variable mass having the following form:

$$L = m(\tau)(g_{\epsilon\sigma} - \gamma_{\epsilon\sigma}) U^{\epsilon} \frac{\underline{\nabla}\Psi^{\sigma}}{\underline{\nabla}\tau} + S_{\epsilon\sigma}\, \frac{\underline{\nabla}\Psi^{\epsilon\sigma}}{\underline{\nabla}\tau} + \left(m(\tau)_{,\mu} + \frac{1}{2} R_{\mu\nu\epsilon\sigma}\, S^{\nu\epsilon\sigma}\right) \Psi^{\mu}. \tag{80}$$

By varying the above Lagrangian with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, we get the following path equation

$$\frac{\underline{\nabla}U^{\alpha}}{\underline{\nabla}\tau} = \frac{m(\tau)_{,\sigma}}{m(\tau)} \left((\mathrm{g}^{\alpha\sigma} - \gamma^{\alpha\sigma}) - U^{\alpha}U^{\sigma}\right) + \frac{1}{2} R^{\alpha}_{\ \mu\rho\sigma}\, S^{\mu\rho\sigma}, \tag{81}$$

and,

$$\frac{\underline{\nabla}S^{\alpha\beta}}{\underline{\nabla}\tau} = 0. \tag{82}$$

Using equations (81) and (82), therefore, the equation of spinning motion will take the form

$$\frac{\underline{\nabla} S^{\alpha\beta\gamma}}{\underline{\nabla}\tau} = \left(\frac{m(\tau)_{,\sigma}}{m(\tau)}\left((\mathrm{g}^{\alpha\sigma} - \gamma^{\alpha\sigma}) - U^{\alpha}U^{\sigma}\right) + \frac{1}{2} R^{\alpha}{}_{\mu\rho\sigma}\, S^{\mu\rho\sigma}\right) S^{\beta\gamma}. \quad (83)$$

Therefore, we can get the equation of spin density deviation by applying the relation (21) together with the condition (20)

$$\frac{\underline{\nabla}^2 \Psi^{\alpha\beta\gamma}}{\underline{\nabla}\tau^2} = S^{\rho[\beta\gamma} R^{\alpha]}{}_{\rho\epsilon\sigma}\, U^{\epsilon}\, \Psi^{\sigma} + \left(\left(\frac{m(\tau)_{,\sigma}}{m(\tau)}\left((\mathrm{g}^{\alpha\sigma} - \gamma^{\alpha\sigma}) - U^{\alpha}U^{\sigma}\right) + \frac{1}{2} R^{\alpha}{}_{\mu\rho\sigma}\, S^{\mu\rho\sigma}\right) S^{\beta\gamma}\right)_{;\delta} \Psi^{\delta}$$

$$+ \left(\left(\frac{m(\tau)_{,\sigma}}{m(\tau)}\left((\mathrm{g}^{\alpha\sigma} - \gamma^{\alpha\sigma}) - U^{\alpha}U^{\sigma}\right) + \frac{1}{2} R^{\alpha}{}_{\mu\rho\sigma}\, S^{\mu\rho\sigma}\right) S^{\beta\gamma}\right)_{|\delta} \Psi^{\delta} \quad (84)$$

4.2. Moffat's Approach

The Lagrangian of a spinning motion in case of variable mass, has been suggested to take the form

$$L = m(\tau)\hat{g}_{\epsilon\sigma} U^{\epsilon} \frac{\hat{\nabla}\Psi^{\sigma}}{\hat{\nabla}\tau} + S_{\epsilon\sigma} \frac{\hat{\nabla}\Psi^{\epsilon\sigma}}{\hat{\nabla}\tau} + \left(m(\tau)_{,\mu} + \frac{1}{2}\, \hat{M}_{\sigma\nu\alpha\beta}\, S^{\nu\alpha\beta}\right)\Psi^{\mu}. \quad (85)$$

By varying the above Lagrangian with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, we get

$$\frac{\hat{\nabla} U^{\alpha}}{\hat{\nabla}\tau} = \frac{m(\tau)_{,\sigma}}{m(\tau)}(\hat{g}^{\alpha\sigma} - U^{\alpha}U^{\sigma}) + \frac{1}{2}\hat{M}^{\alpha}{}_{\mu\rho\sigma}\, S^{\mu\rho\sigma}. \quad (86)$$

and,

$$\frac{\hat{\nabla} S^{\alpha\beta}}{\hat{\nabla}\tau} = 0. \quad (87)$$

Using the Weyssenhoff tensor (12), then the equation of motion for spinning fluid can be obtained to have the form:

$$\frac{\hat{\nabla} S^{\alpha\beta\gamma}}{\hat{\nabla}\tau} = \left(\frac{m(\tau)_{,\sigma}}{m(\tau)}(\hat{g}^{\alpha\sigma} - U^{\alpha}U^{\sigma}) + \frac{1}{2}\hat{M}^{\alpha}{}_{\mu\rho\sigma}\, S^{\mu\rho\sigma}\right) S^{\beta\gamma}. \quad (88)$$

Applying (45) and the condition (44) accordingly, we can get the equation of spinning density deviation to become

$$\frac{\hat{\nabla}^2 \Psi^{\alpha\beta\gamma}}{\hat{\nabla}\tau^2} = S^{\rho[\beta\gamma} \hat{M}^{\alpha]}{}_{\rho\epsilon\sigma}\, U^{\epsilon}\, \Psi^{\sigma} + \left(\left(\frac{m(\tau)_{,\sigma}}{m(\tau)}(\hat{g}^{\alpha\sigma} - U^{\alpha}U^{\sigma}) + \frac{1}{2}\hat{M}^{\alpha}{}_{\mu\rho\sigma}\, S^{\mu\rho\sigma}\right) S^{\beta\gamma}\right)_{||\delta} \Psi^{\delta}. \quad (89)$$

4.3. BIMOND Type Theories

We suggest the following Lagrangian to drive equation of motion for spinning fluid in case of variable mass to be written as

$$L = m(\tau)\bar{g}_{\epsilon\sigma}U^{\epsilon}\frac{\overline{\nabla}\Psi^{\sigma}}{\overline{\nabla}\tau} + S_{\epsilon\sigma}\frac{\overline{\nabla}\Psi^{\epsilon\sigma}}{\overline{\nabla}\tau} + \left[m(\tau)_{,\mu} + \frac{1}{2}\left(R_{\mu\nu\epsilon\sigma} - \overline{N}_{\mu\nu\epsilon\sigma}\right)S^{\nu\epsilon\sigma}\right]\Psi^{\mu}. \quad (90)$$

The path equations can be obtained by applying variation with respect to Ψ^{α} *and* $\Psi^{\alpha\beta}$, to (90), as follow:

$$\frac{\overline{\nabla}U^{\alpha}}{\overline{\nabla}\tau} = \frac{m(\tau)_{,\epsilon}}{m(\tau)}(\bar{g}^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}\left(R^{\alpha}{}_{\mu\rho\sigma} - \overline{N}^{\alpha}{}_{\mu\rho\sigma}\right)S^{\mu\rho\sigma}, \quad (91)$$

and,

$$\frac{\overline{\nabla}S^{\alpha\beta}}{\overline{\nabla}\tau} = 0. \quad (92)$$

Using (91) and (92), also, by taking in consideration the Weyssenhoff tensor (12), then the equation of motion can be written as

$$\frac{\overline{\nabla}S^{\alpha\beta\gamma}}{\overline{\nabla}\tau} = \left[\frac{m(\tau)_{,\epsilon}}{m(\tau)}(\bar{g}^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}\left(R^{\alpha}{}_{\mu\rho\sigma} - \overline{N}^{\alpha}{}_{\mu\rho\sigma}\right)S^{\mu\rho\sigma}\right]S^{\beta\gamma}. \quad (93)$$

Following the same rules mentioned above using (20), (21), (44)and (45) to derive the spin density deviation equation, we get

$$\frac{\overline{\nabla}^2\Psi^{\alpha\beta\gamma}}{\overline{\nabla}\tau^2} = \left(S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma} - S^{\rho[\beta\gamma}\overline{N}^{\alpha]}{}_{\rho\epsilon\sigma}\right)U^{\epsilon}\Psi^{\sigma} + \left(\left[\frac{m(\tau)_{,\epsilon}}{m(\tau)}(\bar{g}^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}\left(R^{\alpha}{}_{\mu\rho\sigma} - \overline{N}^{\alpha}{}_{\mu\rho\sigma}\right)S^{\mu\rho\sigma}\right]S^{\beta\gamma}\right)_{;\delta}\Psi^{\delta}$$
$$+ \left(\left[\frac{m(\tau)_{,\epsilon}}{m(\tau)}(\bar{g}^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}\left(R^{\alpha}{}_{\mu\rho\sigma} - \overline{N}^{\alpha}{}_{\mu\rho\sigma}\right)S^{\mu\rho\sigma}\right]S^{\beta\gamma}\right)_{\underset{+}{|}\delta}\Psi^{\delta}. \quad (94)$$

4.4. Bi-metric Theories

We proposed a Lagrangian able to describe the motion for matter and twin matter in case of variable mass having the following form:

$$L = m(\tau)g_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma}{}_{;\alpha}U^{\alpha} + \tilde{m}(\lambda)h_{\epsilon\sigma}V^{\epsilon}\Phi^{\sigma}{}_{|||\alpha}V^{\alpha} + S_{\epsilon\sigma}\Psi^{\epsilon\sigma}{}_{;\alpha}U^{\alpha} + \tilde{S}_{\epsilon\sigma}\Phi^{\epsilon\sigma}{}_{|||\alpha}V^{\alpha}$$
$$+ \left(m(\tau)_{,\sigma} + \frac{1}{2m(\tau)}R_{\sigma\nu\alpha\beta}S^{\nu\alpha\beta}\right)\Psi^{\mu} + \left(\tilde{m}(\lambda)_{,\sigma} + \frac{1}{2}\tilde{R}_{\sigma\nu\alpha\beta}\tilde{S}^{\nu\alpha\beta}\right)\Phi^{\mu}. \quad (95)$$

Taking the variation with respect to Ψ^{α}, Φ^{α} ,$\Psi^{\alpha\beta}$ *and* $\Phi^{\alpha\beta}$, to (95), we obtain the following set of equations:

$$\left.\begin{aligned}\frac{DU^{\alpha}}{D\tau} &= \frac{m(\tau)_{,\nu}}{m(\tau)}(g^{\mu\nu} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}R^{\alpha}_{\ \mu\rho\sigma}\, S^{\nu\rho\sigma}.\\ \frac{DV^{\alpha}}{D\lambda} &= \frac{\tilde{m}(\lambda)_{,\epsilon}}{\tilde{m}(\lambda)}(h^{\alpha\epsilon} - V^{\alpha}V^{\epsilon}) + \frac{1}{2}\tilde{R}^{\alpha}_{\ \nu\rho\sigma}\, \tilde{S}^{\nu\rho\sigma}.\\ \frac{DS^{\alpha\beta}}{D\tau} &= 0.\\ \frac{D\tilde{S}^{\alpha\beta}}{D\lambda} &= 0.\end{aligned}\right\} \quad (96)$$

Using the Weyssenhoff tensor (12), then the equations of motion can be written as

$$\frac{DS^{\alpha\beta\gamma}}{D\tau} = \left(\frac{m(\tau)_{,\epsilon}}{m(\tau)}(g^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}R^{\alpha}_{\ \mu\rho\sigma}\, S^{\nu\rho\sigma}\right) S^{\beta\gamma}. \quad (97)$$

$$\frac{D\tilde{S}^{\alpha\beta\gamma}}{D\lambda} = \left(\frac{\tilde{m}(\lambda)_{,\epsilon}}{\tilde{m}(\lambda)}(h^{\alpha\epsilon} - V^{\alpha}V^{\epsilon}) + \frac{1}{2}\tilde{R}^{\alpha}_{\ \nu\rho\sigma}\, \tilde{S}^{\nu\rho\sigma}\right) \tilde{S}^{\beta\gamma}. \quad (98)$$

Applying relations (21) and (59) together with conditions (20) and (60), we get the following set of equations of spin density deviation

$$\frac{D^2\Psi^{\alpha\beta\gamma}}{D\tau^2} = S^{\rho[\beta\gamma}R^{\alpha]}_{\ \ \rho\epsilon\sigma}\, U^{\epsilon}\, \Psi^{\sigma} + \left(\left(\frac{m(\tau)_{,\epsilon}}{m(\tau)}(g^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}R^{\alpha}_{\ \mu\rho\sigma}\, S^{\nu\rho\sigma}\right) S^{\beta\gamma}\right)_{;\delta} \Psi^{\delta}. (99)$$

$$\frac{D^2\Phi^{\alpha\beta\gamma}}{D\lambda^2} = \tilde{S}^{\rho[\beta\gamma}\tilde{R}^{\alpha]}_{\ \ \rho\epsilon\sigma}\, V^{\epsilon}\, \Phi^{\sigma} + \left(\left(\frac{\tilde{m}(\lambda)_{,\epsilon}}{\tilde{m}(\lambda)}(h^{\alpha\epsilon} - V^{\alpha}V^{\epsilon}) + \frac{1}{2}\tilde{R}^{\alpha}_{\ \nu\rho\sigma}\, \tilde{S}^{\nu\rho\sigma}\right) \tilde{S}^{\beta\gamma}\right)_{|||\delta} \Phi^{\delta}. (100)$$

4.5. Bi-metric invariant-gravitation theory: Verozub Approach

The Lagrangian function for a spinning motion in case of variable mass in this version can be expressed as:

$$L = m(\tau)\Upsilon_{\epsilon\sigma}(\psi)U^{\epsilon}\frac{\underline{D}\Psi^{\sigma}}{\underline{D}\tau} + S_{\epsilon\sigma}\frac{\underline{D}\Psi^{\epsilon\sigma}}{\underline{D}\tau} + \left(m(\tau)_{,\sigma} + \frac{1}{2}K_{\mu\nu\epsilon\sigma}\, S^{\nu\epsilon\sigma}\right)\Psi^{\mu}. \quad (101)$$

By operating the Euler-Lagrangian equations (15) and (15) to (101), we get:

$$\frac{\underline{D}U^{\alpha}}{\underline{D}\tau} = \frac{m(\tau)_{,\epsilon}}{m(\tau)}(\Upsilon^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}K^{\alpha}_{\ \nu\epsilon\sigma}\, S^{\nu\epsilon\sigma}, \quad (102)$$

and,

$$\frac{\underline{D}S^{\alpha\beta}}{\underline{D}\tau} = 0. \qquad (103)$$

Using equations (102) and (103), accordingly, the equation of spinning motion can be written as

$$\frac{\underline{D}S^{\alpha\beta\gamma}}{\underline{D}\tau} = \left(\frac{m(\tau)_{,\epsilon}}{m(\tau)}(\Upsilon^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}K^{\alpha}_{\ \nu\epsilon\sigma}\, S^{\nu\epsilon\sigma}\right) S^{\beta\gamma}. \qquad (104)$$

The equation of spin density deviation can be derived using relation (74) together with condition (73), as

$$\frac{\underline{D}^2\Psi^{\alpha\beta\gamma}}{\underline{D}\tau^2} = S^{\rho[\beta\gamma}R^{\alpha]}_{\ \rho\epsilon\sigma}\, U^{\epsilon}\, \Psi^{\sigma} + \left(\left(\frac{m(\tau)_{,\epsilon}}{m(\tau)}(\Upsilon^{\alpha\epsilon} - U^{\alpha}U^{\epsilon}) + \frac{1}{2}K^{\alpha}_{\ \nu\epsilon\sigma}\, S^{\nu\epsilon\sigma}\right) S^{\beta\gamma}\right)_{\underset{+}{||}\delta} \Psi^{\delta}. \qquad (105)$$

5. Equations of Motion for Spinning Charged Fluids and its corresponding spin deviation

In this Section, we are going to demonstrate the equation of motion for spinning charged fluids in case of $P^{\epsilon} \neq mU^{\epsilon}$.

5.1. Rosen's Approach

The Lagrangian function for charged spinning fluid can be suggested to have the form

$$L = (g_{\epsilon\sigma} - \gamma_{\epsilon\sigma})P^{\epsilon}\frac{\underline{\nabla}\Psi^{\sigma}}{\underline{\nabla}\tau} + S_{\epsilon\sigma}\,\frac{\underline{\nabla}\Psi^{\epsilon\sigma}}{\underline{\nabla}\tau} + \frac{1}{2}R_{\mu\nu\epsilon\sigma}\, S^{\epsilon\sigma}U^{\mu}\Psi^{\nu} + 2P_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}$$

$$+q\, F_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma} + \frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma;\upsilon}\Psi^{\upsilon} + \frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma|\upsilon}\Psi^{\upsilon} - \left(M_{\epsilon\rho}F^{\rho}_{\ \sigma} + M_{\sigma\rho}F^{\rho}_{\ \epsilon}\right)\Psi^{\epsilon\sigma}. \qquad (106)$$

Operating the Euler- Lagrangian equations (15) and (16), to the Lagrangian (106), we get:

$$\frac{\underline{\nabla}P^{\alpha}}{\underline{\nabla}\tau} = \frac{1}{2}R^{\alpha}_{\ \mu\rho\sigma}\, S^{\rho\sigma}U^{\mu} + q\, F^{\alpha}_{\ \epsilon}U^{\epsilon} + \frac{1}{2}g^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\alpha} + \frac{1}{2}\gamma^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|\mu}, \qquad (107)$$

and,

$$\frac{\underline{\nabla}S^{\alpha\beta}}{\underline{\nabla}\tau} = 2P^{[\alpha}U^{\beta]} - \left(M^{\alpha\rho}F_{\rho}^{\ \beta} + M^{\beta\rho}F_{\rho}^{\ \alpha}\right). \qquad (108)$$

Meanwhile, from Weyssenhoff tensor (10), we can obtain the equation of motion for a charged spinning fluid as follow:

$$\frac{\underline{\nabla}S^{\alpha\beta\gamma}}{\underline{\nabla}\tau} = P^{\alpha}\left[2P^{[\beta}U^{\gamma]} - \left(M^{\beta\rho}F_{\rho}^{\ \gamma} + M^{\gamma\rho}F_{\rho}^{\ \beta}\right)\right] + \left[\frac{1}{2}R^{\alpha}_{\ \mu\rho\sigma}\, S^{\rho\sigma}U^{\mu} + q\, F^{\alpha}_{\ \epsilon}U^{\epsilon}\right.$$

$$+\frac{1}{2}g^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\alpha}+\frac{1}{2}\gamma^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|\mu}\Big]S^{\beta\gamma} \quad (109)$$

Accordingly, we can obtain the equation of spin density deviation by applying the relation (21) and the condition taking the following form

$$\frac{\underline{\nabla}^2\Psi^{\alpha\beta\gamma}}{\underline{\nabla}\tau^2}=S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma}\,U^{\epsilon}\,\Psi^{\sigma}+\left(\left(2P^{[\beta}U^{\gamma]}-\left(M^{\beta\rho}F_{\rho}{}^{\gamma}+M^{\gamma\rho}F_{\rho}{}^{\beta}\right)\right)P^{\alpha}\right)_{;\delta}\Psi^{\delta}$$

$$+\left(\left(2P^{[\beta}U^{\gamma]}-\left(M^{\beta\rho}F_{\rho}{}^{\gamma}+M^{\gamma\rho}F_{\rho}{}^{\beta}\right)\right)P^{\alpha}\right)_{|\delta}\Psi^{\delta}$$

$$+\Bigg[\left(\left(\frac{1}{2}R^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}+\ +\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma;\alpha}+\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma|\alpha}\right)S^{\beta\gamma}\right)_{;\delta}$$

$$+\left(\left(\frac{1}{2}R^{\alpha}{}_{\mu\epsilon\sigma}\,U^{\mu}\,S^{\epsilon\sigma}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}+\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma|\alpha}\ +\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma;\alpha}\right)S^{\beta\gamma}\right)_{|\delta}\Bigg]\Psi^{\delta}. \quad (110)$$

5.2. Moffat's Approach

We suggest the Lagrangian representing charged spinning fluid to have the form

$$L=\hat{g}_{\epsilon\sigma}P^{\epsilon}\frac{\hat{\nabla}\Psi^{\sigma}}{\hat{\nabla}\tau}+S_{\epsilon\sigma}\,\frac{\hat{\nabla}\Psi^{\epsilon\sigma}}{\hat{\nabla}\tau}+\frac{1}{2}\hat{L}_{\mu\nu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\nu}\Psi^{\mu}+2P_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}$$

$$+q\,F_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma}+\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma||\upsilon}\Psi^{\upsilon}-\left(M_{\epsilon\rho}F^{\rho}{}_{\sigma}+M_{\sigma\rho}F^{\rho}{}_{\epsilon}\right)\Psi^{\epsilon\sigma}. \quad (111)$$

By taking the variation with respect to Ψ^{α} *and* $\Psi^{\alpha\beta}$, to the Lagrangian (111), we obtain:

$$\frac{\hat{\nabla}P^{\alpha}}{\hat{\nabla}\tau}=\frac{1}{2}\hat{L}^{\alpha}{}_{\mu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\mu}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}+\frac{1}{2}\hat{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma||\mu}, \quad (112)$$

and

$$\frac{\hat{\nabla}S^{\alpha\beta}}{\hat{\nabla}\tau}=2P^{[\alpha}U^{\beta]}-\left(M^{\alpha\rho}F_{\rho}{}^{\beta}+M^{\beta\rho}F_{\rho}{}^{\alpha}\right). \quad (113)$$

Therefore, using the Weyssenhoff tensor (10), the equation of spinning motion for a charged spinning fluid will have the form

$$\frac{\hat{\nabla}S^{\alpha\beta\gamma}}{\hat{\nabla}\tau}\ =2P^{\alpha}P^{[\beta}U^{\gamma]}-\left(M^{\beta\rho}F_{\rho}{}^{\gamma}+M^{\gamma\rho}F_{\rho}{}^{\beta}\right)+\frac{1}{2}\hat{L}^{\alpha}{}_{\mu\epsilon\sigma}S^{\epsilon\sigma}\,S^{\beta\gamma}U^{\mu}$$

$$+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}S^{\beta\gamma}+\frac{1}{2}\hat{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma||\mu}S^{\beta\gamma}. \quad (114)$$

The equation of spin density deviation can be obtained using relation (45) together with conditions (44), as follow

$$\frac{\hat{\nabla}^2 \Psi^{\alpha\beta\gamma}}{\hat{\nabla}\tau^2} = S^{\rho[\beta\gamma}\hat{L}^{\alpha]}{}_{\rho\epsilon\sigma}\, U^{\epsilon}\,\Psi^{\sigma} + \Big(2P^{\alpha}P^{[\beta}U^{\gamma]} - \Big(M^{\beta\rho}F_{\rho}{}^{\gamma} + M^{\gamma\rho}F_{\rho}{}^{\beta}\Big)$$

$$+\frac{1}{2}\hat{L}^{\alpha}_{\ \mu\epsilon\sigma}S^{\epsilon\sigma}\,S^{\beta\gamma}U^{\mu} + q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}S^{\beta\gamma} + \frac{1}{2}\hat{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma||\mu}S^{\beta\gamma}\Big)_{||\delta}\Psi^{\delta}. (115)$$

5.3. BIMOND Type Theories

The Lagrangian of charged spinning fluid, can be suggested to have the form

$$L = \bar{g}_{\epsilon\sigma}U^{\epsilon}\frac{\overline{\nabla}\Psi^{\sigma}}{\overline{\overline{\nabla}}\tau} + S_{\epsilon\sigma}\,\frac{\overline{\nabla}\Psi^{\epsilon\sigma}}{\overline{\overline{\nabla}}\tau} + \frac{1}{2}\left(R_{\mu\nu\epsilon\sigma} - \overline{N}_{\mu\nu\epsilon\sigma}\right)S^{\epsilon\sigma}U^{\nu}\Psi^{\mu} + 2\bar{P}_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}$$

$$+q\,F_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma} + \frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma;\nu}\Psi^{\upsilon} + \frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma|^{\nu}_{+}}\Psi^{\upsilon} - \left(M_{\epsilon\rho}F^{\rho}_{\ \sigma} + M_{\sigma\rho}F^{\rho}_{\ \epsilon}\right)\Psi^{\epsilon\sigma}. (116)$$

The path equations can be obtained by applying variation with respect to Ψ^{α} *and* $\Psi^{\alpha\beta}$, to the Lagrangian (121), we get:

$$\frac{\overline{\nabla}P^{\alpha}}{\overline{\overline{\nabla}}\tau} = \frac{1}{2}\left(R^{\alpha}_{\ \mu\epsilon\sigma} - \overline{N}^{\alpha}_{\ \mu\epsilon\sigma}\right)S^{\epsilon\sigma}U^{\mu} + q\,F^{\alpha}_{\ \epsilon}U^{\epsilon} + \frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu} + \frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|^{\mu}_{+}}. (117)$$

and,

$$\frac{\overline{\nabla}S^{\alpha\beta}}{\overline{\overline{\nabla}}\tau} = 2P^{[\alpha}U^{\beta]} - \Big(M^{\alpha\rho}F_{\rho}{}^{\beta} + M^{\beta\rho}F_{\rho}{}^{\alpha}\Big). \qquad (118)$$

Meanwhile, the equation of motion for a charged spinning fluid using the Weyssenhoff tensor (10), can be expressed as.

$$\frac{\overline{\nabla}S^{\alpha\beta\gamma}}{\overline{\overline{\nabla}}\tau} = 2P^{\alpha}P^{[\beta}U^{\gamma]} - P^{\alpha}\Big(M^{\beta\rho}F_{\rho}{}^{\gamma} + M^{\gamma\rho}F_{\rho}{}^{\beta}\Big) + \frac{1}{2}\left(R^{\alpha}_{\ \mu\epsilon\sigma} - \overline{N}^{\alpha}_{\ \mu\epsilon\sigma}\right)S^{\epsilon\sigma}\,S^{\beta\gamma}U^{\mu}$$

$$+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}S^{\beta\gamma} + \frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}S^{\beta\gamma} + \frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|^{\mu}_{+}}S^{\beta\gamma}. \qquad (119)$$

Using the same rules mentioned above using (20), (21), (44) and (45) to derive the spin density deviation equation, we obtain

$$\frac{\overline{\nabla}^2\Psi^{\alpha\beta\gamma}}{\overline{\nabla}\tau^2} = \Big(S^{\rho[\beta\gamma}R^{\alpha]}{}_{\rho\epsilon\sigma} - S^{\rho[\beta\gamma}\overline{N}^{\alpha]}{}_{\rho\epsilon\sigma}\Big)U^{\epsilon}\,\Psi^{\sigma}$$

$$+\left(2\bar{P}^{\alpha}\bar{P}^{[\beta}U^{\gamma]} - P^{\alpha}\Big(M^{\beta\rho}F_{\rho}{}^{\gamma} + M^{\gamma\rho}F_{\rho}{}^{\beta}\Big)\right)_{;\delta}\Psi^{\delta}$$

$$+\left(2\bar{P}^{\alpha}\bar{P}^{[\beta}U^{\gamma]}-P^{\alpha}\left(M^{\beta\rho}F_{\rho}^{\ \gamma}+M^{\gamma\rho}F_{\rho}^{\ \beta}\right)\right)_{|^{\delta}_{+}}\Psi^{\delta}$$

$$+\left(\left(\frac{1}{2}R^{\alpha}_{\ \mu\epsilon\sigma}\,U^{\mu}S^{\epsilon\sigma}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}+\frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}\right.\right.$$

$$\left.\left.+\frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|^{\mu}_{+}}S^{\beta\gamma}\right)S^{\beta\gamma}\right)_{;\delta}\Psi^{\delta}+\left(\left(\frac{1}{2}\bar{N}^{\alpha}_{\ \mu\epsilon\sigma}\,U^{\mu}S^{\epsilon\sigma}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}++\frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}\right.\right.$$

$$\left.\left.+\frac{1}{2}\bar{g}^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|^{\mu}_{+}}S^{\beta\gamma}\right)S^{\beta\gamma}\right)_{|^{\delta}_{+}}\Psi^{\delta} \qquad (120)$$

5.4. **Bi-metric Theories**

It has been proposed that the Lagrangian of charged spinning fluid describing matter and twin matter has the form:

$$L=g_{\epsilon\sigma}P^{\epsilon}\Psi^{\sigma}_{\ ;\alpha}U^{\alpha}+h_{\epsilon\sigma}\tilde{P}^{\epsilon}\Phi^{\sigma}_{\ |||\alpha}V^{\alpha}+S_{\epsilon\sigma}\,\Psi^{\epsilon\sigma}_{\ ;\alpha}U^{\alpha}+\tilde{S}_{\epsilon\sigma}\,\Phi^{\epsilon\sigma}_{\ |||\alpha}V^{\alpha}+2P_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}$$
$$+\frac{1}{2}R_{\mu\nu\epsilon\sigma}S^{\epsilon\sigma}U^{\nu}\Psi^{\mu}+\frac{1}{2}\tilde{R}_{\mu\nu\epsilon\sigma}\tilde{S}^{\epsilon\sigma}V^{\nu}\Phi^{\mu}+2\tilde{P}_{[\epsilon}V_{\sigma]}\Phi^{\epsilon\sigma}+q\,F_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma}+\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma:\upsilon}\Psi^{\upsilon}$$
$$-\left(M_{\epsilon\rho}F^{\rho}_{\ \sigma}+M_{\sigma\rho}F^{\rho}_{\ \epsilon}\right)\Psi^{\epsilon\sigma}+q\,F_{\epsilon\sigma}V^{\epsilon}\Phi^{\sigma}+\frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma|||\upsilon}\Phi^{\upsilon}$$
$$-\left(M_{\epsilon\rho}F^{\rho}_{\ \sigma}+M_{\sigma\rho}F^{\rho}_{\ \epsilon}\right)\Phi^{\epsilon\sigma}.\,(121)$$

By operating the variation with respect to Ψ^{α} and Φ^{α} $\Psi^{\alpha\beta}$ and $\Phi^{\alpha\beta}$, to the Lagrangian (121), we get the following set of equations

$$\left.\begin{aligned}
\frac{DP^{\alpha}}{D\tau}&=\frac{1}{2}R^{\alpha}_{\ \mu\rho\sigma}\,S^{\rho\sigma}U^{\mu}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}+\frac{1}{2}g^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu},\\
\frac{D\tilde{P}^{\alpha}}{D\lambda}&=\frac{1}{2}\tilde{R}^{\alpha}_{\ \mu\rho\sigma}\,\tilde{S}^{\rho\sigma}V^{\mu}+q\,F^{\alpha}_{\ \epsilon}V^{\epsilon}+\frac{1}{2}h^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|||\mu},\\
\frac{DS^{\alpha\beta}}{D\tau}&=2P^{[\alpha}U^{\beta]}-\left(M^{\alpha\rho}F_{\rho}^{\ \beta}+M^{\beta\rho}F_{\rho}^{\ \alpha}\right),\\
\frac{D\tilde{S}^{\alpha\beta}}{D\lambda}&=2\tilde{P}^{[\alpha}V^{\beta]}-\left(M^{\alpha\rho}F_{\rho}^{\ \beta}+M^{\beta\rho}F_{\rho}^{\ \alpha}\right).
\end{aligned}\right\} \qquad (122)$$

Using the Weyssenhoff tensor (10), then we get the equations of motion for spinning fluid as follow:

$$\frac{DS^{\alpha\beta\gamma}}{D\tau}=P^{\alpha}\left(2P^{[\beta}U^{\gamma]}-\left(M^{\beta\rho}F_{\rho}^{\ \gamma}+M^{\gamma\rho}F_{\rho}^{\ \beta}\right)\right)+\left(\frac{1}{2}R^{\alpha}_{\ \mu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\mu}+q\,F^{\alpha}_{\ \epsilon}U^{\epsilon}\right.$$
$$\left.+\frac{1}{2}g^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}\right)S^{\beta\gamma}. \qquad (123)$$

$$\frac{D\tilde{S}^{\alpha\beta\gamma}}{D\lambda}=\tilde{P}^{\alpha}\left(2\tilde{P}^{[\beta}V^{\gamma]}-\left(M^{\beta\rho}F_{\rho}^{\ \gamma}+M^{\gamma\rho}F_{\rho}^{\ \beta}\right)\right)$$
$$+\left(\frac{1}{2}\,\tilde{R}^{\alpha}_{\ \mu\epsilon\sigma}\,\tilde{S}^{\epsilon\sigma}\,V^{\mu}+q\,F^{\alpha}_{\ \epsilon}V^{\epsilon}+\frac{1}{2}h^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|||\mu}\right)\tilde{S}^{\beta\gamma}. \qquad (124)$$

Applying relations (21) and (60) together with conditions (20) and (59), we can get equations of spin density deviation

$$\frac{D^2\Psi^{\alpha\beta\gamma}}{D\tau^2} = S^{\rho[\beta\gamma}R^{\alpha]}_{\ \ \rho\epsilon\sigma}U^{\epsilon}\,\Psi^{\sigma} + \Big[\tilde{P}^{\alpha}\big(2\tilde{P}^{[\beta}V^{\gamma]} - M^{\beta\rho}F_{\rho}^{\ \gamma} - M^{\gamma\rho}F_{\rho}^{\ \beta}\big)$$

$$+\left(\frac{1}{2}R^{\alpha}_{\ \mu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\mu} + q\,F^{\alpha}_{\ \epsilon}U^{\epsilon} + \frac{1}{2}g^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}\right)S^{\beta\gamma}\Big]_{;\delta}\Psi^{\delta}$$

$$\frac{D^2\Phi^{\alpha\beta\gamma}}{D\lambda^2} = S^{\rho[\beta\gamma}R^{\alpha]}_{\ \ \rho\epsilon\sigma}U^{\epsilon}\,\Phi^{\sigma} + \Big[\tilde{P}^{\alpha}\Big(2\tilde{P}^{[\beta}V^{\gamma]} - \Big(M^{\beta\rho}F_{\rho}^{\ \gamma} + M^{\gamma\rho}F_{\rho}^{\ \beta}\Big)\Big)$$

$$+\left(\frac{1}{2}\,\tilde{R}^{\alpha}_{\ \mu\epsilon\sigma}\,\tilde{S}^{\epsilon\sigma}\,V^{\mu} + q\,F^{\alpha}_{\ \epsilon}V^{\epsilon} + \frac{1}{2}h^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma|||\mu}\right)\tilde{S}^{\beta\gamma}\Big]_{|||\delta}\Phi^{\delta}. \quad (125)$$

5.5. Bi-metric invariant-gravitation theory: Verozub Approach

We propose the Lagrangian of charged spinning fluid according to this version, to take the form

$$L = \Upsilon_{\epsilon\sigma}(\psi)\check{P}^{\epsilon}\frac{\underline{D}\Psi^{\sigma}}{\underline{D}\tau} + S_{\epsilon\sigma}\,\frac{\underline{D}\Psi^{\epsilon\sigma}}{\underline{D}\tau} + \frac{1}{2}K_{\mu\nu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\nu}\Psi^{\mu} + 2\check{P}_{[\epsilon}U_{\sigma]}\Psi^{\epsilon\sigma}$$

$$+q\,F_{\epsilon\sigma}U^{\epsilon}\Psi^{\sigma} + \frac{1}{2}M^{\epsilon\sigma}F_{\epsilon\sigma||^{\nu}_{+}}\Psi^{\upsilon} - \big(M_{\epsilon\rho}F^{\rho}_{\ \sigma} + M_{\sigma\rho}F^{\rho}_{\ \epsilon}\big)\Psi^{\epsilon\sigma}. \quad (126)$$

Varying the above Lagrangian with respect to Ψ^{α} and $\Psi^{\alpha\beta}$, we get

$$\frac{\underline{D}\check{P}^{\alpha}}{\underline{D}\tau} = \frac{1}{2}K^{\alpha}_{\ \mu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\mu} + q\,F^{\alpha}_{\ \epsilon}U^{\epsilon} + \frac{1}{2}\Upsilon^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}, \quad (127)$$

and,

$$\frac{\underline{D}S^{\alpha\beta}}{\underline{D}\tau} = 2\check{P}^{[\alpha}U^{\beta]} - \Big(M^{\alpha\rho}F_{\rho}^{\ \beta} + M^{\beta\rho}F_{\rho}^{\ \alpha}\Big). \quad (128)$$

Consequently, we can get the equation of motion for spinning fluid using the Weyssenhoff tensor (10) as follow:

$$\frac{\underline{D}S^{\alpha\beta\gamma}}{\underline{D}\tau} = \check{P}^{\alpha}\Big(2\check{P}^{[\beta}U^{\gamma]} - \Big(M^{\beta\rho}F_{\rho}^{\ \gamma} + M^{\gamma\rho}F_{\rho}^{\ \beta}\Big)\Big)$$

$$+\left(\frac{1}{2}K^{\alpha}_{\ \mu\epsilon\sigma}\,S^{\epsilon\sigma}U^{\mu} + q\,F^{\alpha}_{\ \epsilon}U^{\epsilon} + \frac{1}{2}\Upsilon^{\alpha\mu}M^{\epsilon\sigma}F_{\epsilon\sigma;\mu}\right)\,S^{\beta\gamma}. \quad (129)$$

Applying (74) and the condition (73) consequently, it is easy to derive the equation of spinning density deviation, as follow

$$\frac{\underline{D}^2\Psi^{\alpha\beta\gamma}}{\underline{D}\tau^2} = S^{\rho[\beta\gamma}K^{\alpha]}_{\ \ \rho\epsilon\sigma}U^{\epsilon}\,\Psi^{\sigma} + \Big[\check{P}^{\alpha}\Big(2\check{P}^{[\beta}U^{\gamma]} - \Big(M^{\beta\rho}F_{\rho}^{\ \gamma} + M^{\gamma\rho}F_{\rho}^{\ \beta}\Big)\Big)$$

6. Discussion

Equations of motion for spinning fluids in different versions of bi-metric theories are obtained. The basic idea for developing such a type of equations is the ability to examine several cases for particles orbiting strong fields. The problem of motion is vital to test the behaviour of different particles starting from microscopic to macroscopic objects.

Due to this wide spectrum, the usual notation of finding test particle as a probe to examine the viability of any gravitational field theory becomes irrelevant. Accordingly, the demand to find equations of motion for objects having some intrinsic properties such spinning, charged and spinning charged ones led authors to replace it for examining the stability problem for objects orbing strong gravitational fields [32].

Moreover, in order to examine through an insightful vision the behaviour of particles in strong fields, it becomes mandatory to obtain equations of motion for spinning fluids as they act an active role in the accretion disk orbiting the active galactic nucleic like the supermassive black hole SgrA*. From this perspective we have obtain equations of spinning fluids in different types of Bi-metric theories of gravity as in equations (19), (26), (31), (38), (43), (50), (57), (58), (64), (65), (71) and (78) as well as their corresponding deviation equations. We also have extended this work to study some intrinsic properties of the fluid like the case of variable mass which may give an account to reveal the puzzle of dark matter nearby strong fields of gravity. Not only this but also, we have obtained the equations of charged spinning fluids in strong gravitational fields of gravity for different versions of bi metric theories as explained in Equations (109), (114), (123), (124) and (129) as well as their corresponding deviation equations (110), (115), (120), (125) and (130).

Such a finding will be become a glimpse to examine the behaviour of plasma physics in strong gravitational fields which will be studied in our future work.

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