Two Different Functions of A Variable-Order Fractional Derivative for A Vector Host Model

Article · January 2021

DOI: 10.36632/csi/2019.8.4.30

Amaal Sayed
Nile University

Mohamed Khalil
October University for Modern Sciences and Arts

Current Science International **Volume : 08 | Issue : 04| Oct.- Dec. | 2019**
EISSN:2706-7920 ISSN: 2077-4435 **Pages: 911-920**
DOI: 10.36632/csi/2019.8.4.30

Two Different Functions of A Variable-Order Fractional Derivative for A Vector Host Model

Amaal Sayed
Department of Mathematics, Faculty of Engineering and Applied Sciences, Nile University, Giza, Egypt. E-mail: amsayed@nu.edu.eg
M. Khalil
Department of Mathematics, Faculty of Engineering, October University for modern sciences and arts (MSA), Giza, Egypt. E-mail: mkibrahim@msa.eun.eg

A.M.A. Arafa
[2]Department of Mathematics, Faculty of Science, Port Said University, Port Said, Egypt
E-mail: anas_arafa@sci.psu.edu.eg

Received: 30 Sept. 2019 / Accepted 20 Nov. 2019 / Publication date: 30 Dec. 2019

ABSTRACT

Fractional calculus is one of the best mathematical tools to characterize the memory property of the dynamical systems. So, Variable-order fractional derivatives in Caputo sense are used in characterizing the memory property of the presented dynamic a vector host model. Numerical solutions are obtained using a predictor-corrector method to handle the fractional derivatives. These numerical solutions show that the modeling of vector host model with variable-order fractional derivatives has more advantaged than integer-order derivatives.

Keywords: Fractional calculus, dynamical systems, Predictor-corrector method, Variable-order fractional derivatives, and vector host model.

Introduction

The memory property has been found in complex systems (Guerrero *et al.,* 2011). Characterizing the memory property of systems is a challenging issue in phenomenological description. So, fractional calculus is used to characterize the memory property of complex systems (Turski *et al.,* 2002). The variable-order fractional derivative has the significant feature of capturing the history of the variable which cannot be easily done in case of the integer order derivative (Pinto and Machado, 2013). The variable-order fractional derivatives has been presented in many researches(El-Sayed *et al.,* 2016; El-Sayed *et al.,* 2017; Khalil *et al.,* 2018; Arafa *et al.,* 2019) which is good at depicting the memory property which changes with time.

The vector host model is a mathematical model for the spread of a disease that transfers from human to another human by a carrier (vector). The host is the living being that virus, bacteria causing microorganism normally resides in, for example, some bird's species are normal hosts to arboviruses such as West Nile virus. The vector is an organism that helps transmit infection from one host to another. For example, the mosquito serves as the vector to infect humans with the West Nile virus (Marquardt, 2004).

From mathematicians' perspective, mathematical models are significant tools that help us to understand the current state and the future progress of diseases in human networks in order to control and prevent such diseases (Arafa *et al.,* 2019).

This paper is organized as follows. In Section 2, some preliminaries of fractional calculus, the algorithm of the predictor-corrector method are presented, and the proposed model. In Section 3 is devoted to the numerical results and discussions. Our conclusion is illustrated in Section 4.

Corresponding Author: Amaal Sayed, Department of Mathematics, Faculty of Engineering and Applied Sciences, Nile University, Giza, Egypt. E-mail: amsayed@nu.edu.eg

Materials and methods

1. Preliminaries

We present some basic definitions of variable fractional-order derivatives as follows:

Definition 1 (Riemann–Liouville derivatives of variable-order fractional $\alpha(t)$)
Let $\alpha(t)$ be a bounded and continuous function, then Riemann–Liouville variable-order fractional derivative of $f(t)\colon [a,b] \to \mathbb{R}$ is defined as (Xu and He, 2013):

i) Left Riemann–Liouville derivative of variable-order fractional $\alpha(t)$ is defined by

$$ {}_{a}^{RL}D_{t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_{a}^{t} (t-\omega)^{-\alpha(t)} f(\omega) d\omega, \quad 0 < \alpha(t) \le 1 \tag{1} $$

ii) Right Riemann–Liouville derivative of variable-order fractional $\alpha(t)$ is defined by

$$ {}_{t}^{RL}D_{b}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_{t}^{b} (t-\omega)^{-\alpha(t)} f(\omega) d\omega, \quad 0 < \alpha(t) \le 1 \tag{2} $$

Definition 2 (Caputo derivatives of variable-order fractional $\alpha(t)$)
Let $\alpha(t)$ be a bounded and continuous function, then the Caputo variable-order fractional derivative of $f(t)\colon [a,b] \to \mathbb{R}$ is defined as (Xu and He, 2013). :

i) Left Caputo derivative of variable-order fractional $\alpha(t)$ is defined by

$$ {}_{a}^{C}D_{t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_{a}^{t} (t-\omega)^{-\alpha(t)} f'(\omega) d\omega, \quad 0 < \alpha(t) \le 1 \tag{3} $$

ii) Right Caputo derivative of variable-order fractional $\alpha(t)$ is defined by

$$ {}_{t}^{C}D_{b}^{\alpha(t)} f(t) = \frac{-1}{\Gamma(1-\alpha(t))} \int_{t}^{b} (t-\omega)^{-\alpha(t)} f'(\omega) d\omega, \quad 0 < \alpha(t) \le 1 \tag{4} $$

2. The model derivation

The vector-host model consists of two different populations, human population and vector population. The human population is divided into three different subgroups, susceptible $x_h(t)$, infected $y_h(t)$ and recovered $z_h(t)$. The vector population is divided into two different subgroups, susceptible $x_v(t)$, and infected $y_v(t)$. We convert the model which is presented in (ELmojtaba, 2016) into variable-order fractional model as follows:

$$ \begin{aligned} {}^{C}D^{\alpha_1(t)} x_h(t) &= -ab x_h(t) y_v(t), \\ {}^{C}D^{\alpha_2(t)} y_h(t) &= ab x_h(t) y_v(t) - \beta y_h(t), \\ {}^{C}D^{\alpha_3(t)} z_h(t) &= \beta y_h(t), \\ {}^{C}D^{\alpha_4(t)} x_v(t) &= -ac x_v(t) y_h(t), \\ {}^{C}D^{\alpha_4(t)} y_v(t) &= ac x_v(t) y_h(t). \end{aligned} \tag{5} $$

The parameters of the model are

- a is the per capita biting rate.
- b is the transmission probability per bite per human.
- c is the transmission probability for vector infection.
- β is the average rate of infected humans recover and acquire permanent immunity.

a. The predictor-corrector method

There are many techniques for solving a variable-order fractional model. We state a predictor-corrector method for solving a variable-order fractional model.

We will introduce an algorithm of predictor-corrector method for solving the following system of variable-order fractional differential equations

$$
\begin{aligned}
D^{\alpha_1(t)}x_h(t) &= f_1(x_h(t), y_h(t), z_h(t), x_v(t), y_v(t)),\\
D^{\alpha_2(t)}y_h(t) &= f_2(x_h(t), y_h(t), z_h(t), x_v(t), y_v(t)), \qquad 0 \le t \le T\\
D^{\alpha_3(t)}z_h(t) &= f_3(x_h(t), y_h(t), z_h(t), x_v(t), y_v(t)),\\
D^{\alpha_4(t)}x_v(t) &= f_4(x_h(t), y_h(t), z_h(t), x_v(t), y_v(t)),\\
D^{\alpha_5(t)}y_v(t) &= f_4(x_h(t), y_h(t), z_h(t), x_v(t), y_v(t)),
\end{aligned} \tag{6}
$$

With $0 < \alpha_i(t) \le 1 (i = 1,2,3,4,5)$ and initial conditions $(x_h(0), y_h(0), z_h(0), x_v(0), y_v(0))$.

Evaluate the predicted values as follows:

$$
\begin{aligned}
(x_h)_{n+1}^p &= x_h(0) + \sum_{j=0}^{n} \frac{\beta_{1,j,n+1}}{\Gamma(\alpha_1(t_{n+1}))} f_1((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),\\
(y_h)_{n+1}^p &= y_h(0) + \sum_{j=0}^{n} \frac{\beta_{2,j,n+1}}{\Gamma(\alpha_2(t_{n+1}))} f_2((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),\\
(z_h)_{n+1}^p &= z_h(0) + \sum_{j=0}^{n} \frac{\beta_{3,j,n+1}}{\Gamma(\alpha_3(t_{n+1}))} f_3((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),\\
(x_v)_{n+1}^p &= x_v(0) + \sum_{j=0}^{n} \frac{\beta_{4,j,n+1}}{\Gamma(\alpha_4(t_{n+1}))} f_4((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),\\
(y_v)_{n+1}^p &= y_v(0) + \sum_{j=0}^{n} \frac{\beta_{5,j,n+1}}{\Gamma(\alpha_5(t_{n+1}))} f_5((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j).
\end{aligned} \tag{7}
$$

Where

$$
\beta_{i,j,n+1} = \frac{h^{\alpha_i(t_{n+1})}}{\alpha_i(t_{n+1})}[(n-j+1)^{\alpha_i(t_{n+1})} - (n-j)^{\alpha_i(t_{n+1})}].\ h = T/N,\ T_n = nh. \tag{8}
$$

EISSN: 2706-7920 ISSN: 2077-4435 **DOI: 10.36632/csi/2019.8.4.30**

Evaluate the corrected values as follows

$$(x_h)_{n+1}=x_h(0)+\frac{h^{\alpha_1(t_{n+1})}}{\Gamma(\alpha_1(t_{n+1})+2)}+f_1((x_h)^p_{n+1},(y_h)^p_{n+1},(z_h)^p_{n+1},(x_v)^p_{n+1})+\sum_{j=0}^{n}\frac{h^{\alpha_1(t_{n+1})}\gamma_{1,j,n+1}}{\Gamma(\alpha_1(t_{n+1}))+2)}f_1((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),$$

$$(y_h)_{n+1}=y_h(0)+\frac{h^{\alpha_2(t_{n+1})}}{\Gamma(\alpha_2(t_{n+1})+2)}+f_2((x_h)^p_{n+1},(y_h)^p_{n+1},(z_h)^p_{n+1},(x_v)^p_{n+1})+\sum_{j=0}^{n}\frac{h^{\alpha_2(t_{n+1})}\gamma_{2,j,n+1}}{\Gamma(\alpha_2(t_{n+1}))+2)}f_2((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),$$

$$(z_h)_{n+1}=z_h(0)+\frac{h^{\alpha_3(t_{n+1})}}{\Gamma(\alpha_3(t_{n+1})+2)}+f_3((x_h)^p_{n+1},(y_h)^p_{n+1},(z_h)^p_{n+1},(x_v)^p_{n+1})+\sum_{j=0}^{n}\frac{h^{\alpha_3(t_{n+1})}\gamma_{3,j,n+1}}{\Gamma(\alpha_3(t_{n+1}))+2)}f_3((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),$$

$$(x_v)_{n+1}=x_v(0)+\frac{h^{\alpha_4(t_{n+1})}}{\Gamma(\alpha_4(t_{n+1})+2)}+f_4((x_h)^p_{n+1},(y_h)^p_{n+1},(z_h)^p_{n+1},(x_v)^p_{n+1})+\sum_{j=0}^{n}\frac{h^{\alpha_4(t_{n+1})}\gamma_{4,j,n+1}}{\Gamma(\alpha_4(t_{n+1}))+2)}f_4((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j),$$

$$(y_v)_{n+1}=y_v(0)+\frac{h^{\alpha_4(t_{n+1})}}{\Gamma(\alpha_5(t_{n+1})+2)}+f_5((x_h)^p_{n+1},(y_h)^p_{n+1},(z_h)^p_{n+1},(x_v)^p_{n+1})+\sum_{j=0}^{n}\frac{h^{\alpha_5(t_{n+1})}\gamma_{5,j,n+1}}{\Gamma(\alpha_5(t_{n+1}))+2)}f_5((x_h)_j,(y_h)_j,(z_h)_j,(x_v)_j,(y_v)_j).$$

where

$$\gamma_{i,j,n+1}=\begin{cases} n^{\alpha_i(t_{n+1})+1}-(n-\alpha_i(t_{n+1}))(n+1)^{\alpha_i(t_{n+1})} & ,\ j=0,\\ (n-j-2)^{\alpha_i(t_{n+1})+1}+(n-j)^{\alpha_i(t_{n+1})+1}-2(n-j+1)^{\alpha_i(t_{n+1})} & ,\ 1\le j\le n,\\ 1 & ,\ j=n+1 \end{cases} \tag{10}$$

Results and Discussion

We applied the predictor-corrector method to get the numerical solution of the system (5) with the values of the parameters

$a=0.01, b=0.2, c=0.2, \beta=0.3.$

and initial conditions $x_h(0)=100, y_h(0)=6, z_h(0)=1, x_v(0)=80, y_v(0)=12.$

we investigate the system behavior in two cases. First case when the variable-order fractional is $\alpha_i(t)=0.8-0.005t$. Second case when the variable-order fractional $\alpha_i(t)=0.8-0.05sin(\pi t)$.

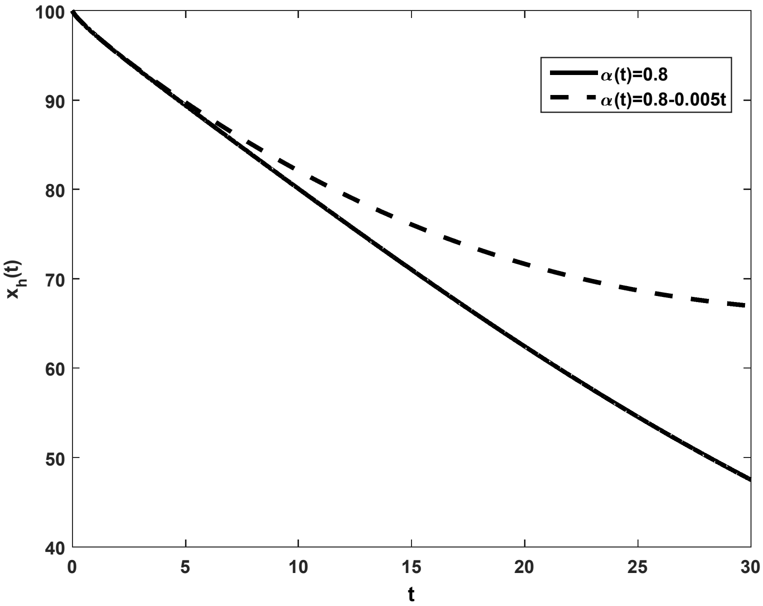

Fig. 1: he solution of $x_h(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = 0.8 - 0.005t$ (dashed line).

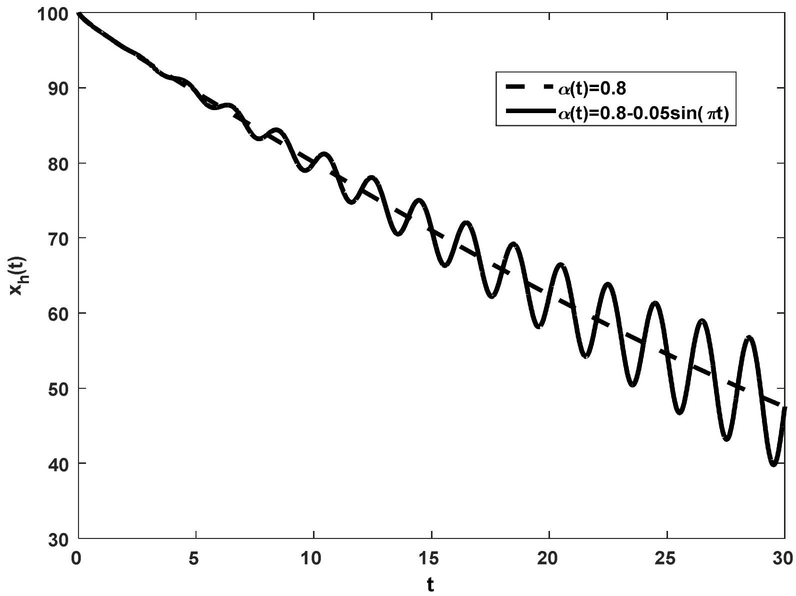

Fig. 2: The solution of $x_h(t)$ at $\alpha(t) = 0.8$ (the dashed line) and at $\alpha(t) = 0.8 - 0.05 sin(\pi t)$ (solid line).

Curr. Sci. Int., 8(4): 911-920, 2019
EISSN: 2706-7920 ISSN: 2077-4435 **DOI: 10.36632/csi/2019.8.4.30**

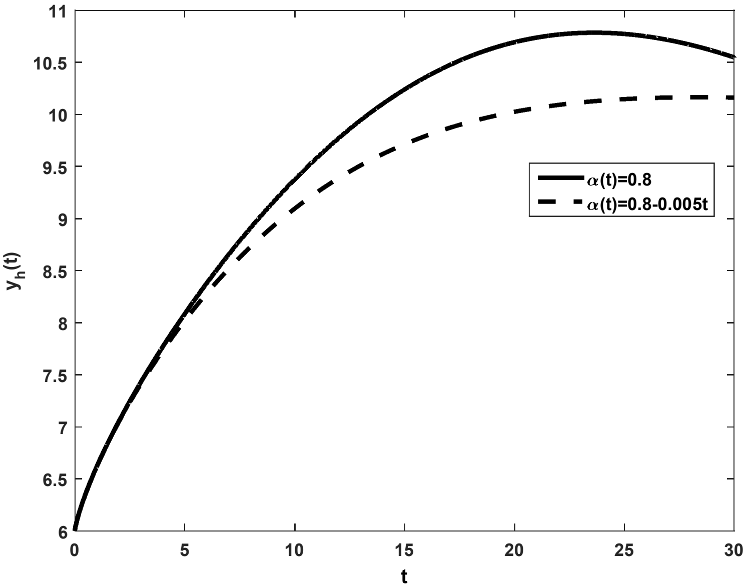

Fig. 3: The solution of y_h(t) at $\alpha(t)=0.8$ (the solid line) and at $\alpha(t)=0.8-0.005t$ (dashed line).

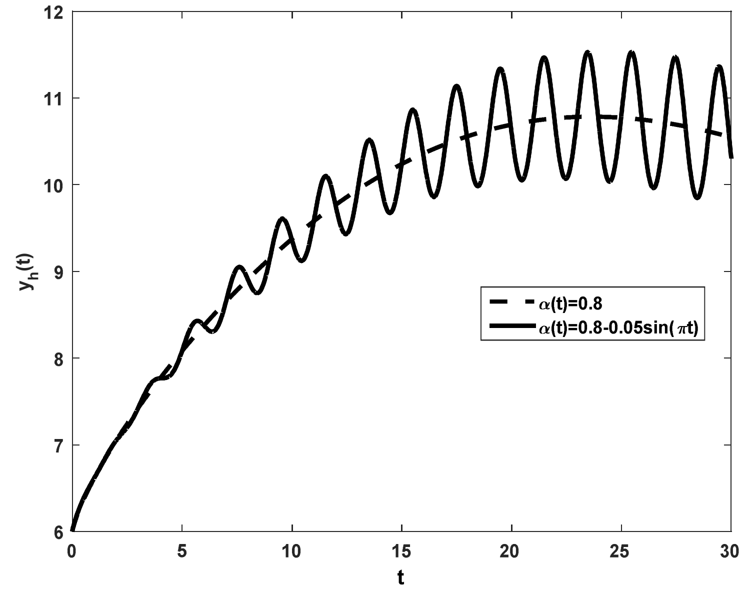

Fig. 4: The solution of y_h(t) at $\alpha(t)=0.8$ (the dashed line) and at $\alpha(t)=0.8-0.05sin(\pi t)$ (solid line).

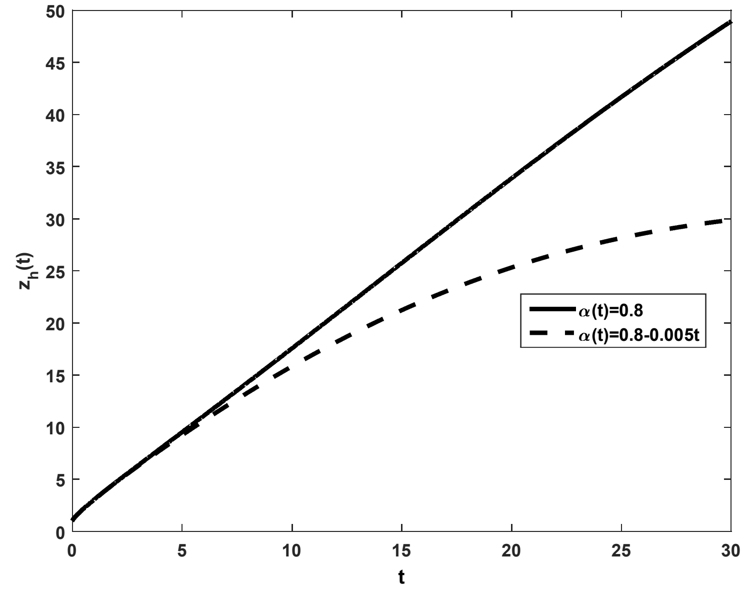

Fig. 5: The solution of z_h(t) at $\alpha(t)=0.8$ (the solid line) and at $\alpha(t)=0.8-0.005t$ (dashed line).

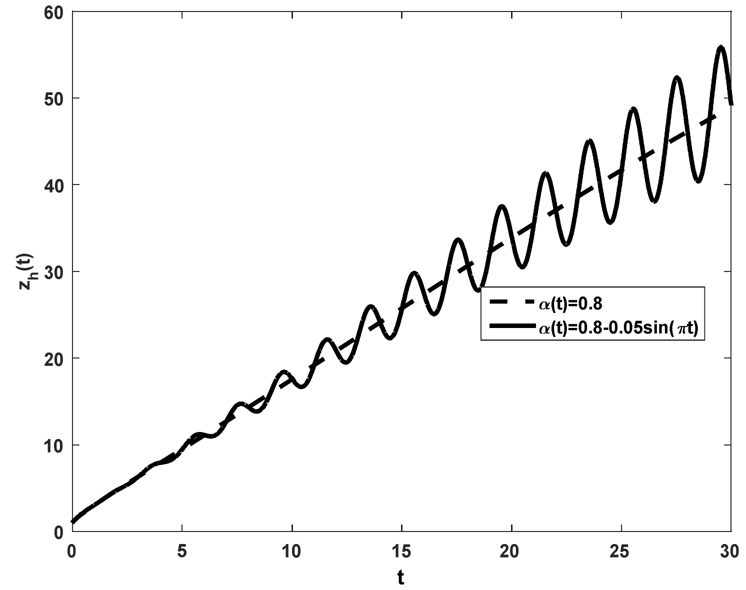

Fig. 6: The solution of z_h(t) at $\alpha(t)=0.8$ (the dashed line) and at $\alpha(t)=0.8-0.05sin(\pi t)$ (solid line).

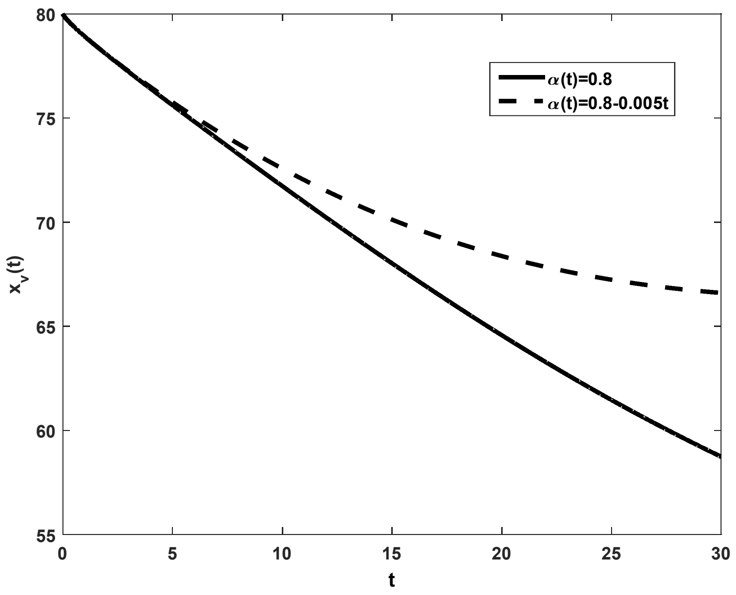

Fig. 7: The solution of $x_v(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = 0.8 - 0.005t$ (dashed line).

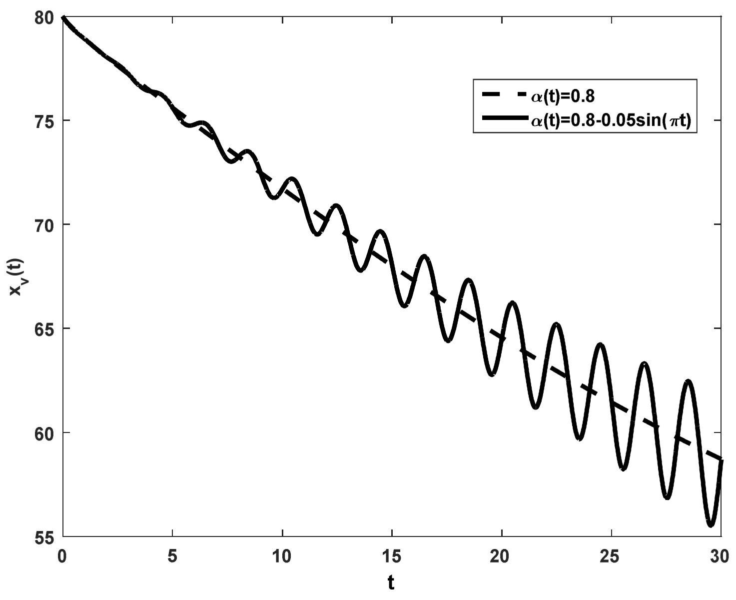

Fig. 8: The solution of $x_v(t)$ at $\alpha(t) = 0.8$ (the dashed line) and at $\alpha(t) = 0.8 - 0.05 sin(\pi t)$ (solid line).

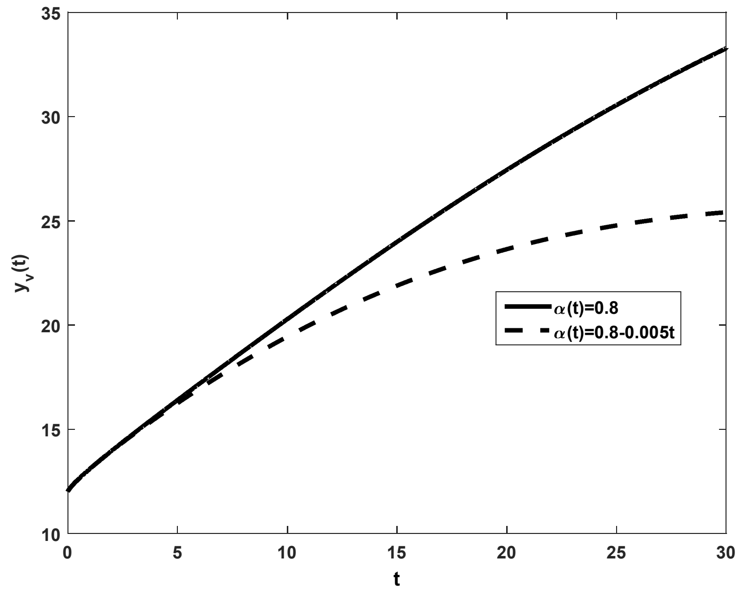

Fig. 9: The solution of $y_v(t)$ at $\alpha(t) = 0.8$ (the solid line) and at $\alpha(t) = 0.8 - 0.005t$ (dashed line).

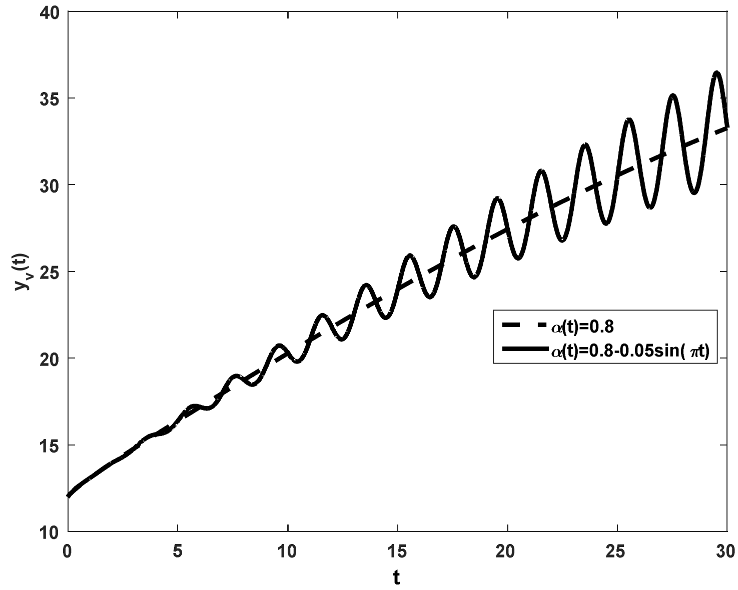

Fig. 10: The solution of $y_v(t)$ at $\alpha(t) = 0.8$ (the dashed line) and at $\alpha(t) = 0.8 - 0.05sin(\pi t)$ (solid line).

In Figs. 1, 3, 5, 7, 9 we take the variable-order fractional $\alpha_i(t) = 0.8 - 0.005t$ means the memory of the system is a decreasing function so the system behavior is slower with time.

In figs. 2, 4, 6, 8, 10 we take the variable-order fractional $\alpha_i(t) = 0.8 - 0.05sin(\pi t)$ means the memory of the system is a periodic function so the system behavior is a periodic.

EISSN: 2706-7920 ISSN: 2077-4435 **DOI: 10.36632/csi/2019.8.4.30**

Conclusion

This paper introduced two different functions of variable-order fractional $\alpha(t)$ for a vector host model. We used the numerical results to show that according to the formula of the memory of the system such as a decreasing function or a periodic function the behavior of the system has the same properties of the memory function.

References

Arafa, A.A.M., M. Khalil and A. Sayed, 2019. A Non-Integer Variable Order Mathematical Model of Human Immunodeficiency Virus and Malaria Co-infection with Time Delay" complexity, Hindawi, article ID 4291017.

ELmojtaba, I.M., 2016. Using Adomian Decomposition Method for Solving a Vector-Host Model. International Journal of Applied 5(2):107-109.

El-Sayed, A.M.A., A.A. Arafa, M. Khalil and Amaal Sayed, 2017. A. Backward Bifurcation in a Fractional Order Epidemiological Model". An international journal of progress in fractional differentiation and application, 3(4):281-287.

El-Sayed, A.M.A., M. Khalil, A.A.M. Arafa and Amaal Sayed, 2016. Numerical behavior of a fractional order dynamical model of RNA silencing". International Journal of Scientific World, 4(2):52-56.

Guerrero, F., F.J. Santonja and R.J. Villanueva, 2011. Analysing the effect of Spanish smoke-free legislation of year 2006: a new method to quantify its impact using a dynamic model, Int. J. Drug Policy, 22, 247–251.

Khalil, M., A. A. Arafa, & Amaal Sayed, 2018. A variable fractional order network model of Zika virus". Journal of Fractional Calculus and Applications, 9(1): 204-221.

Marquardt, William H. Biology of Disease Vectors. 2nd ed. New York: Academic Press, 2004.

Pinto, C.M.A. and J.A.T. Machado, 2013. Fractional model for malaria transmission under control strategies", Computers and Mathematics with Applications, 66(5) 908-916.

Turski, A.J., B. Atamaniuk, and E. Turska. 2002. "Fractional derivative analysis of Helmholtz and paraxial-wave equations". arXiv preprint physics/0203047.

Xu, Y. and Z. He (2013), Existence and uniqueness results for Cauchy problem of variable-order fractional differential equations, J. Appl. Math. Comput., 43(1-2):295-306.