Motion of Spinning and Spinning Deviation Density Tensors in Riemannian Geometry

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Abstract Equations of motion of spinning density for extended objects, and corresponding deviation equations are derived. The problem of motion for a variable mass to a spinning extended object is obtained. Spinning fluids may be considered as a special case to express the motion of spinning density for extended objects. Meanwhile, spinning density tensor can be expressed in terms of tetrad formalism of General Relativity to be regarded as a gauge theory of gravity. Equations of spinning and spinning deviation density tensors have been derived using a specific type of Bazanski Lagrangian is performed.

1 Introduction

Spinning motion is regarded as one of the actual features of the characteristic behavior for objects in nature, which led many authors to focus on the cause of the spinning process. Would be eligible to include its internal properties or discard them as a step of simplification? From this perspective, it is vital to begin with equations of motion of spinning Mathisson-Papapetrou [1]

$$\frac{DP^{\alpha}}{Ds} = \frac{1}{2} R^{\alpha}_{\ \cdot \beta\gamma\delta} S^{\gamma\delta} U^{\beta} \tag{1.1}$$

where P^{α} is the momentum of the particle

$$P^{\alpha} = (mU^{\alpha} + U_{\mu} \frac{DS^{\alpha\mu}}{Ds}),$$

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$R^{\alpha}_{.\beta\gamma\delta}$ is the Riemannian curvature, $U^{\alpha} = \frac{dx^{\alpha}}{ds}$ is the unit tangent vector, s is a parameter varying along the curve and $S^{\gamma\delta}$ is the spin tensor. For a spinning object with precession (Gyroscopic motion) can be described using the following equation

$$\frac{DS^{\mu\nu}}{Ds} = P^{\mu}U^{\nu} - P^{\nu}U^{\mu} \tag{1.2}$$

If $P^{\alpha} = mU^{\alpha}$ then equation (1.1) and (1.2) become, the Papapetrou equation for short!

$$\frac{DU^{\alpha}}{Ds} = \frac{1}{2m} R^{\alpha}_{\beta\gamma\delta} S^{\gamma\delta} U^{\beta}, \tag{1.3}$$

and

$$\frac{DS^{\mu\nu}}{Ds} = 0, \tag{1.4}$$

provided that [2]

$$S^{\mu\nu} = \bar{s}(U^{\mu}\Psi^{\nu} - U^{\nu}\Psi^{\mu}) \tag{1.5}$$

where, $\bar{s}$ spin magnitude, U^{α} is four vector velocity and Ψ^{α} is a geodesic deviation vector. Equation (1.3) can be obtained from geodesic equations [3]

$$\frac{DU^{\alpha}}{D\tau} = 0 \tag{1.6}$$

and the geodesic deviation equations

$$\frac{D^2\Psi^{\alpha}}{D\tau^2} = R^{\alpha}_{\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho}. \tag{1.7}$$

If we apply the following transformation of paths for different parameters [2]

$$\frac{dx^{\alpha}}{ds} = \frac{dx^{\alpha}}{d\tau} + \beta \frac{D\Psi^{\alpha}}{D\tau} \tag{1.8}$$

where $\beta \stackrel{\text{def}}{=} \frac{\bar{s}}{m}$ and by operating covariant derivative with respect to the parameter s on both sides, we get

$$\frac{DU^{\alpha}}{Ds} = \frac{DU^{\alpha}}{D\tau} + \beta \frac{D^2\Psi^{\alpha}}{D\tau^2} \tag{1.9}$$

Using geodesic equations (1.6) and geodesic deviation equations (1.7) as well Equation(1.3) one can obtain Equation (1.1), while if one consider Frenkel condition

$$S_{\mu\nu} U^{\mu} = 0$$

to be covarient differentiated on both sides and after some manipulations one can get Equation(1.4).

Thus, due to the extension of spin tensor form pole-dipole moments to multi-pole moments for extended objects, this may lead to examine its corresponding propagation equation [4].

Such an equation may be obtained by means of introducing the spin tensor density$S^{\alpha\beta\gamma}$ as a third order skew symmetric tensor is viable to be describe extended objects. These equations play a vital role in astrophysics and early cosmology to become a good candidate for describing a spinning fluid and also for describing the status of the accretion disc orbiting a compact gravitational field as in AGN [5]. Also, it contribute to understanding the problem of motion quark-gluon heavy ion collisions in early universe [6].

In our present work we are going to derive equations of spinning density tensor and spin deviation density tensor of different cases as described in Riemannian geometry in GR. Accordingly, on studying the spinning density tensor a stringent relation arises its composition within thermodynamics variables[7].

$$T d\hat{s} = dE + pd(\frac{1}{\rho}) - \frac{1}{2}\omega_{\mu\nu} ds^{\mu\nu} \tag{1.10}$$

where T is the temperature, $\hat{s}$ is the entropy, E is energy density , $\omega_{\mu\nu}$ is the spin angular velocity and $s^{\mu\nu}$ spin density. Thus, in self consistent theories described the entropy becomes conserved i.e. such that

$$\frac{d\hat{s}}{ds} = 0. \tag{1.11}$$

Thus, the first law of thermodynamics becomes

$$\frac{dE}{ds} + p\frac{d\rho^{-1}}{ds} - \frac{1}{2}\omega_{\mu\nu}\frac{ds^{\mu\nu}}{ds} = 0. \tag{1.12}$$

Meanwhile, it is worth to clarify that thermodynamics transports coefficients such as viscosity which is connected with identifying the nature of the spin tensor [8]. It is well known that spinning fluids are dominating properties of nature, may be found to describe the problem of motion of particles in an accretion disc as a Gyrodynamics fluid. This is the counterpart of the Papapetrou equation [9]. Owing to spin density tensor, an interaction between spinning motion and thermodynamics variables may be found in equation (1.12) [10].

Thus, it is well known that $S^{\alpha\beta\gamma}$ is a third order tensor, viable to define extended objects. Nevertheless, from a progenitor case, the spin density tensor is bounded to be a skew symmetric in the last two indices, it comes to arise for expressing a spinning fluid element as a confided case of an extended object. From this perspective, the notation of importing the expression propagating equation for spin density [4] the modified becomes crystal clear to express spin fluids [7]. Yet, one may find out that the Weyssenhoff tensor [6] is most eligible candidate to express a spin fluid element i.e.

$$S^{\rho\mu\nu} = S^{\mu\nu} U^{\rho}. \tag{1.13}$$

Thus, differentiating both sides of (1.13) by the covariant derivative to get

$$\frac{DS^{\rho\mu\nu}}{Ds} = \frac{DS^{\mu\nu}}{Ds}U^{\rho} + \frac{DU^{\rho}}{Ds}S^{\mu\nu}. \tag{1.14}$$

Now, one uses the following Lagrangian [11]

$$L = g_{\mu\nu}U^{\mu}\frac{D\Psi^{\nu}}{Ds} + S_{\mu\nu}\frac{D\Psi^{\nu\nu}}{Ds}. \tag{1.15}$$

By taking the variation with respect to the deviation vector Ψ^{ρ} one gets (1.6),

$$\frac{DU^{\rho}}{Ds} = 0.$$

Also, taking the variation with respect to the spinning deviation tensor $\Psi^{\rho\delta}$ to get (1.4)

$$\frac{DS^{\rho\delta}}{Ds} = 0. \tag{1.16}$$

Consequently, substituting from (1.6) and (1.16) into (1.14), one obtain

$$\frac{DS^{\rho\delta\sigma}}{Ds} = 0, \tag{1.17}$$

which indicates equation of spin density tensor. Accordingly, we may obtain equations analogously by means of its corresponding Bazanski Lagrangian stemmed from its original formalism [3] and its modification in GR [12] to become

$$L = S_{\alpha\mu\nu}\frac{D\Psi^{\alpha\mu\nu}}{Ds}, \tag{1.18}$$

such that by taking the variation with respect to $\Psi^{\rho\delta\lambda}$,

$$\frac{DS^{\rho\delta\lambda}}{Ds} = 0. \tag{1.19}$$

If we apply the commutation relation in such that

$$(S^{\rho\mu\nu}{}_{;\alpha\beta} - S^{\rho\mu\nu}{}_{;\beta\alpha})U^{\alpha}\Psi^{\beta} = S^{\sigma[\mu\nu}R^{\rho]}{}_{\sigma\alpha\beta}U^{\alpha}\Psi^{\beta}, \tag{1.20}$$

and

$$S^{\rho\mu\nu}{}_{;\delta}\Psi^{\rho} = \Psi^{\rho\mu\nu}{}_{;\delta}U^{\rho}. \tag{1.21}$$

Then we obtain its corresponding spin density deviation tensor equation

$$\frac{D^2\psi^{\rho\delta\lambda}}{Ds^2} = S^{\sigma[\delta\lambda}R^{\rho]}{}_{\sigma\alpha\beta}U^{\alpha}\Psi^{\beta}. \tag{1.22}$$

The importance of spin density deviation tensor is to examine the stability conditions of a spinning density orbiting a compact object [13].

Accordingly, the paper is organized as follows. In Section 2, we discusses spinning density tensor and spinning density deviation tensor equations: Papapetrou-like equations. While Section 3 displays spinning density tensor and spinning density deviation tensor equations: a variable mass. And Section 4 emphasizes the concept of spinning density tensor and spinning density deviation tensor equations : a spinning fluid. Spinning and spinning density deviation equations for a gauge theory of gravity: tetrad formalism is explained in Section 5. Section 6 presents conclusion and future work.

2 Spinning Density Tensor and Spinning Density Deviation Tensor Equations: Papapetrou-Like Equations

In this section, we are going to examine a massive density spin tensor able to describe an orbiting extended object for a compact object. Accordingly, the Weyssenhoff spin vector may be amended to be expressed as follows:

$$\bar{S}^{\rho\mu\nu} = S^{\mu\nu} P^{\rho}, \tag{2.1}$$

where P^{ρ} is the momentum in which it is relating to $\bar{S}^{\rho\mu\nu}$ in the following sense

$$\bar{S}^{\rho\mu\nu} = S^{\mu\nu}(mU^{\rho} + U_{\delta}\frac{DS^{\rho\delta}}{Ds}), \tag{2.2}$$

i.e.

$$\bar{S}^{\rho\mu\nu} = S^{\mu\nu}(mU^{\rho} + U_{\delta}(P^{\rho}U^{\delta} - P^{\delta}U^{\rho})). \tag{2.3}$$

Differentiating both sides by covariant derivative for (2.1) to get

$$\frac{D\bar{S}^{\rho\mu\nu}}{Ds} = \frac{DS^{\mu\nu}}{Ds}P^{\rho} + \frac{DP^{\rho}}{Ds}S^{\mu\nu}. \tag{2.4}$$

We suggest the equivalent Bazanski Lagrangian to be:

$$L = \bar{S}_{\rho\mu\nu}\frac{D\bar{\Psi}^{\rho\mu\nu}}{Ds} + f_{\rho\mu\nu}\bar{\Psi}^{\rho\mu\nu}. \tag{2.5}$$

Thus, by taking the variation with respect to its corresponding deviation tensor $\Psi^{\rho\mu\nu}$, we get

$$\frac{D\bar{S}^{\rho\mu\nu}}{Ds} = f^{\rho\mu\nu}, \tag{2.6}$$

in which

$$f^{\rho\mu\nu} = \frac{DS^{\mu\nu}}{Ds}P^{\rho} + \frac{DP^{\rho}}{Ds}S^{\mu\nu},$$

becomes

$$f^{\rho\mu\nu} = (P^{\mu}U^{\nu} - P^{\nu}U^{\mu})P^{\rho} + \frac{1}{2}R^{\rho}{}_{\alpha\sigma\beta}S^{\alpha\sigma\beta}S^{\mu\nu},$$

where

$$f^{\mu} = \frac{1}{2}R^{\mu}{}_{\nu\rho\delta}S^{\nu\rho\delta},$$

is regarded as a spin force, and

$$M^{\mu\nu} = P^{\mu}U^{\nu} - P^{\nu}U^{\mu}.$$

Consequently, its corresponding spin density deviation tensor equation can be obtained by applying in a similar way the commutation relations as given in (1.21) and (1.22) the corresponding spinning density deviation equation:

$$\frac{D^2\Psi^{\rho\mu\nu}}{Ds^2} = S^{\delta[\mu\nu}\bar{R}^{\rho]}{}_{\delta\alpha\beta}U^{\alpha}\Psi^{\beta} + f^{\rho\mu\nu}{}_{;\delta}\Psi^{\delta}. \tag{2.7}$$

3 Spinning Density Tensor and Spinning Density Deviation Tensor Equations: A Variable Mass

If we consider a massive spin density tensor whose mass is not constant but function of the parameter (s) in which its corresponding Weyssenhoff tensor becomes as follows

$$\hat{S}^{\rho\mu\nu} = m(s)U^{\rho}S^{\mu\nu}. \tag{3.1}$$

Differentiating both sides we obtain:

$$\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = \frac{D(m(s)U^{\rho})}{Ds}S^{\mu\nu} + m(s)U^{\rho}\frac{DS^{\mu\nu}}{Ds}. \tag{3.2}$$

Thus we can suggest a Lagrangian obtained for spinning variable mass object

$$L = m(s)g_{\mu\nu}U^{\mu}\frac{D\Psi^{\nu}}{Ds} + (m(s)_{\rho} + \frac{1}{2}R_{\rho\alpha\beta\gamma}S^{\alpha\beta\gamma})\Psi^{\rho} + S_{\alpha\beta}\frac{D\Psi^{\alpha\beta}}{Ds} + m(s)_{\rho}\Psi^{\rho}. \tag{3.3}$$

Thus, by taking the variation with respect to Ψ^{δ} to get

$$\frac{DU^{\delta}}{Ds} = \frac{m(s)_{\rho}}{m(s)}(g^{\delta\rho} - U^{\delta}U^{\rho}) + \frac{1}{2m(s)}R^{\delta}{}_{\rho\alpha\beta}S^{\rho\alpha\beta}. \tag{3.4}$$

And by taking the variation with respect to $\Psi^{\rho\delta}$ to obtain

$$\frac{D\hat{S}^{\mu\nu}}{Ds} = 0. \tag{3.5}$$

Thus the spinning density tensor with a variable mass may be expressed in the following way:

$$\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = (\frac{m(s)_{\sigma}}{m(s)}(g^{\sigma\rho} - U^{\sigma}U^{\rho}) + \frac{1}{2m(s)}R^{\rho}{}_{\lambda\alpha\beta}S^{\lambda\alpha\beta})S^{\mu\nu}. \tag{3.6}$$

Accordingly, its corresponding Bazanski Lagrangian may be expressed as

$$L = \hat{S}_{\rho\mu\nu}\frac{D\bar{\Psi}^{\rho\mu\nu}}{Ds} + \bar{f}_{\rho\mu\nu}\Psi^{\rho\mu\nu}, \tag{3.7}$$

where $\bar{f}^{\rho\mu\nu} = (\frac{m(s)_{\sigma}}{m(s)}(g^{\sigma\rho} - U^{\sigma}U^{\rho}) + \frac{1}{2m(s)}R^{\rho}{}_{\lambda\alpha\beta}S^{\lambda\alpha\beta})S^{\mu\nu}$. By taking the variation with respect to $\Psi^{\rho\sigma\delta}$ we obtain

$$\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = \bar{f}^{\rho\mu\nu}, \tag{3.8}$$

As we follow the same procedure as given in equation (1.21) we obtain the deviation spinning density equation to become by taking the variation with respect to $\Psi^{\rho\sigma\delta}$ we obtain

$$\frac{D^2\Psi^{\rho\mu\nu}}{Ds^2} = (\bar{f}^{\rho\mu\nu})_{;\delta}\Psi^{\delta} + S^{\xi[\mu\nu}R^{\rho]}{}_{\xi\alpha\beta}U^{\alpha}\Psi^{\beta}. \tag{3.9}$$

This equation may be applied in studying the stability of a variable spinning disk which may work to explain the effect of mass excess in a region orbiting a compact object. Such an illustration may give rise to examine the effect of dark matter in the accretion disk of a compact object.

4 Spinning Density Tensor and Spinning Density Deviation Tensor Equations: A Spinning Fluid

In this section, we are going to suggest that the relation between a variable mass and a spinning fluid in the following way,

$$\hat{S}^{\rho\mu\nu} = (p + \rho)(s)U^{\rho}S^{\mu\nu}, \tag{4.1}$$

where, $m(s) = (p + \rho)$.

Accordingly, if the pressure is turning to be only parameter of s, while the density is becoming constant. Owing to this suggestion the spin fluid behaves like a spinning variable mass, such that:

$$\frac{dm(s)}{dx^p} = \frac{dp(s)}{dx^p}. \tag{4.2}$$

Such an equivalence may give rise to suggest the following Lagrangian,

$$L = (p + \rho)(s)g_{\mu\nu}U^{\mu}\frac{D\Psi^{\nu}}{Ds} + (P_{\rho} + \frac{1}{2}R_{\rho\alpha\beta\gamma}S^{\beta\gamma}U^{\alpha})\Psi^{\rho} + S_{\alpha\beta}\frac{D\Psi^{\alpha\beta}}{Ds} + P_{\rho}\Psi^{\rho}. \tag{4.3}$$

Such that, by taking the variation with respect to Ψ^{δ} to get

$$\frac{DU^{\delta}}{Ds} = \frac{p_{\rho}}{(p+\rho)}(g^{\delta\rho} - U^{\delta}U^{\rho}) + \frac{1}{2(p+\rho)}R^{\delta}_{\rho\alpha\beta}S^{\rho\alpha\beta}. \tag{4.4}$$

Also, by taking the variation with respect to $\Psi^{\rho\delta}$ to obtain

$$\frac{D\hat{S}^{\mu\nu}}{Ds} = 0 \tag{4.5}$$

Equation of a spinning fluid can be expressed in the following way:

$$\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = (\frac{p_{\sigma}}{(p+\rho)}(g^{\sigma\rho} - U^{\sigma}U^{\rho}) + \frac{1}{2(p+\rho)}R^{\rho}{}_{\lambda\alpha\beta}S^{\lambda\alpha\beta})S^{\mu\nu}. \tag{4.6}$$

Accordingly, its corresponding Bazanski Lagrangian may be expressed as

$$L = \hat{S}_{\rho\mu\nu}\frac{D\bar{\Psi}^{\rho\mu\nu}}{Ds} + \bar{f}_{\rho\mu\nu}\Psi^{\rho\mu\nu}, \tag{4.7}$$

where, $\bar{f}^{\rho\mu\nu} = (\frac{p_{\sigma}}{p+\rho}(g^{\sigma\rho} - U^{\sigma}U^{\rho}) + \frac{1}{2(p+\rho)}R^{\rho}{}_{\lambda\alpha\beta}S^{\lambda\alpha\beta})S^{\mu\nu}$. By taking the variation with respect to $\Psi^{\rho\sigma\delta}$ we obtain

$$\frac{D\hat{S}^{\rho\mu\nu}}{Ds} = \bar{f}^{\rho\mu\nu}, \tag{4.8}$$

By taking the variation with respect to $\Psi^{\rho\sigma\delta}$ we obtain

$$\frac{D^2\Psi^{\rho\mu\nu}}{Ds^2} = (\bar{f}^{\rho\mu\nu})_{;\delta}\Psi^{\delta} + S^{\xi[\mu\nu}R^{\rho]}{}_{\xi\alpha\beta}U^{\alpha}\Psi^{\beta}. \tag{4.9}$$

4.1 Modified Forms of Spin Density

In this part, we suggest a modified form of spin density tensor in Riemannian geometry to be;

(a) <u>$P^{\alpha} = mU^{\alpha}$</u>

$$S^{\alpha\beta\gamma} = \frac{1}{3!}(S^{\beta\gamma}U^{\alpha} + S^{\gamma\alpha}U^{\beta} + S^{\alpha\beta}U^{\gamma}). \tag{4.10}$$

Differentiating both sides using covariant derivative,

$$\frac{DS^{\alpha\beta\gamma}}{Ds} = \frac{1}{3!}(\frac{DS^{\beta\gamma}}{Ds}U^{\alpha} + S^{\beta\gamma}\frac{DU^{\alpha}}{Ds} + \frac{S^{\gamma\alpha}}{Ds}U^{\beta} + S^{\gamma\alpha}\frac{DU^{\beta}}{Ds} + \frac{DS^{\alpha\beta}}{Ds}U^{\gamma} + S^{\alpha\beta}\frac{DU^{\gamma}}{Ds}). \tag{4.11}$$

(i) For $\frac{DU^{\alpha}}{Ds} = 0$ and $\frac{DS^{\alpha\beta}}{Ds} = 0$ and substituting in (4.11), we obtain

$$\frac{DS^{\alpha\beta\gamma}}{Ds} = 0. \tag{4.12}$$

(ii) For $\frac{DU^{\alpha}}{Ds} = \frac{1}{2m}R^{\alpha}{}_{\beta\gamma\delta}S^{\gamma\delta}U^{\alpha}$ and $\frac{DS^{\alpha\beta}}{Ds} = 0$, (4.11) can be rewritten as

$$\frac{DS^{\alpha\beta\gamma}}{Ds} = \frac{1}{3!}(\frac{1}{2m}R^{\alpha}{}_{\mu\nu\rho}S^{\nu\rho}U^{\mu}S^{\beta\gamma} + \frac{1}{2m}R^{\beta}{}_{\mu\nu\rho}S^{\nu\rho}U^{\mu}S^{\gamma\alpha} + \frac{1}{2m}R^{\gamma}{}_{\mu\nu\rho}S^{\nu\rho}U^{\mu}S^{\alpha\beta}), \tag{4.13}$$

i.e.

$$\frac{DS^{\alpha\beta\gamma}}{Ds} = \frac{1}{3!}(\frac{1}{2}(R^{\alpha}_{\ \mu\nu\rho}S^{\beta\gamma} + R^{\beta}_{\ \mu\nu\rho}S^{\gamma\alpha} + R^{\gamma}_{\ \mu\nu\rho}S^{\alpha\beta})S^{\mu\nu\rho}), \tag{4.14}$$

to become

$$\frac{DS^{\alpha\beta\gamma}}{Ds} = \frac{1}{2m}(R^{(\alpha}_{\ \mu\nu\rho}S^{\beta\gamma)})S^{\mu\nu\rho}, \tag{4.15}$$

(b)$\underline{P^{\alpha} \neq mU^{\alpha}}$

$$S^{\alpha\beta\gamma} = \frac{1}{3!}(S^{\beta\gamma}P^{\alpha} + S^{\gamma\alpha}P^{\beta} + S^{\alpha\beta}P^{\gamma}). \tag{4.16}$$

Differentiating both sides using covariant derivative,

$$\frac{D\bar{S}^{\alpha\beta\gamma}}{Ds} = \frac{1}{3!}(\frac{DS^{\beta\gamma}}{Ds}P^{\alpha} + S^{\beta\gamma}\frac{DP^{\alpha}}{Ds} + \frac{DS^{\gamma\alpha}}{Ds}P^{\beta} + S^{\gamma\alpha}\frac{DP^{\beta}}{Ds} + \frac{DS^{\alpha\beta}}{Ds}P^{\gamma} + S^{\alpha\beta}\frac{DP^{\gamma}}{Ds}). \tag{4.17}$$

However, if $\frac{DP^{\alpha}}{Ds} = \frac{1}{2}R^{\alpha}_{\ \beta\gamma\delta}S^{\gamma\delta}U^{\beta}$ and $\frac{DS^{\alpha\beta}}{Ds} = P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha}$ then, we obtain

$$\frac{D\bar{S}^{\alpha\beta\gamma}}{Ds} = \frac{1}{3!}((P^{\beta}U^{\gamma} - P^{\gamma}U^{\beta})P^{\alpha} + \frac{1}{2}R^{\alpha}_{\ \mu\nu\rho}S^{\mu\nu\rho}S^{\beta\gamma} + (P^{\gamma}U^{\alpha} - P^{\alpha}U^{\gamma})P^{\beta}$$
$$+ \frac{1}{2}R^{\beta}_{\ \mu\nu\rho}S^{\mu\nu\rho}S^{\gamma\alpha} + (P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha})P^{\gamma} + \frac{1}{2}R^{\gamma}_{\ \mu\nu\rho}S^{\mu\nu\rho}S^{\alpha\beta}), \tag{4.18}$$

in which can be

$$\frac{D\bar{S}^{\alpha\beta\gamma}}{Ds} = \frac{1}{12}S^{\mu\nu\sigma}(S^{\alpha\beta}R^{\gamma}_{\ \mu\nu\sigma} + S^{\beta\gamma}R^{\alpha}_{\ \mu\nu\sigma} + S^{\gamma\alpha}R^{\beta}_{\ \mu\nu\sigma}). \tag{4.19}$$

To get their corresponding deviation equation as similar as equation (4.9).

5 Spinning and spinning density deviation equations for a gauge theory of gravity : A Tetrad Formalism

5.1 A class of Gauge theories of gravity in Riemannian geometry

The problem of studying microscopic structure of particles led many relativists to get a paradigm shift towards gauge theories of gravity. From this perspective, the building blocks of this type of theories stands for an analogy between the quantities used in current gauge theories and space time. It is of interest to revisit primarily GR to be descriptive by gauge theory and their counterpart in non-Riemannian geometry [14, 15]. Yet, collins et al (1989) described GR as a gauge theory of gravity subject to the following gauge potential vectors e^{a}_{μ} and $\omega^{ij}_{\cdot\,\cdot\,\mu}$ to represent translational and rotational gauge potentials respectively. From this perspective, the problem of invariance of any quantities must be a covariant derivative invariant under general coordinate transformation (GCT) and Local Lorentz transformation (LLT) that are expressible in terms of gauge potential of translation and rotation in the following way [16]. The equations of physics will contain derivatives of tensor fields and it is therefore necessary to define the covariant derivatives of tensor fields under the transformations GCT and LLT, one must need to define two types of connection fields to be associated with each of them. Accordingly, the Christoffel symbol $\left\{ {\alpha \atop \mu\nu} \right\}$ is referred to GCT while the spin connection $\omega_{ab\mu}$ as related to LLT. Accordingly ,

$$D_\mu e^m_\nu \stackrel{\text{def}}{=} e^m_{\nu,\mu} - \{ {\alpha \atop \nu\mu} \} e^m_\alpha + \omega_{\mu}{}^{.m}_{.\ n} e^n_\nu. \tag{5.1}$$

Provided that

$$D_\rho D_\mu e^m_\nu = D_\mu D_\rho e^m_\nu. \tag{5.2}$$

Using this concept it turns out that GR may be expressed in terms of connecting e^μ_a , $\omega_{ab\mu}$ and $\left\{ {\alpha \atop \mu\nu} \right\}$ together

$$g_{\mu\nu} \stackrel{\text{def}}{=} e^a_\mu e^b_\nu \eta_{ab}, \tag{5.3}$$

$$\{ {\alpha \atop \mu\nu} \} \stackrel{\text{def}}{=} \frac{1}{2} g^{\alpha\sigma}(g_{\nu\sigma,\alpha} + g_{\sigma\alpha,\nu} - g_{\alpha\nu,\sigma}). \tag{5.4}$$

And for any arbitrary vector A_μ

$$A_{\mu;cd} - A_{\mu;dc} = R^\alpha_{.\mu dc} A_\alpha, \tag{5.5}$$

where $R^\alpha_{.\mu dc}$ the curvature tensor may be defined, due to gauge approach, in terms of spin connection $\omega^a{}_{b\mu}$

$$R^c_{.d\mu\nu} \stackrel{\text{def}}{=} \omega^c{}_{d\nu,\mu} - \omega^c{}_{d\mu,\nu} + \omega^c{}_{a\mu}\omega^a{}_{d\nu} - \omega^c{}_{a\nu}\omega^a{}_{d\mu}$$

From this perspective, one can figure out the effect of both mesh indices and world indices on describing the curvature tensor.

5.2 Spinning and Spinning Deviation equation of an object in a class of gauge theory in Riemannian geometry

In a similar way to [16] one can find out the equations of motion for a spinning object with precession for a class of gauge theory by taking the variation with respect to Ψ^μ and $\Psi^{\mu\nu}$ simultaneously, for the following Lagrangian

$$L = g_{\mu\nu} P^\mu \frac{D\Psi^\nu}{Ds} + \frac{1}{2} R_{\mu\nu ab} S^{ab} U^\nu \Psi^\mu + e^a_\mu e^b_\nu S_{ab} \frac{D\Psi^{\mu\nu}}{Ds} + (P_\mu U_\nu - P_\nu U_\mu)\Psi^{\mu\nu}, \tag{5.6}$$

to obtain

$$\frac{DP^\alpha}{Ds} = \frac{1}{2} R^\alpha{}_{\nu ab} S^{ab} U^\nu. \tag{5.7}$$

Multiply both sides by e^c_α

$$\frac{DP^c}{Ds} = \frac{1}{2} R^c{}_{\nu ab} S^{ab} U^\nu, \tag{5.8}$$

and

$$\frac{DS^{\alpha\beta}}{Ds} = (P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha}), \tag{5.9}$$

i.e.

$$\frac{De^{a}_{\alpha}e^{b}_{\beta}S^{ab}}{Ds} = (P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha}), \tag{5.10}$$

$$e^{a}_{\alpha}e^{b}_{\beta}\frac{DS^{ab}}{Ds} + S^{ab}\frac{D(e^{a}_{\alpha}e^{b}_{\beta})}{Ds} = (P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha}), \tag{5.11}$$

multiplying both sides by $e^{c}_{\alpha}e^{d}_{\beta}$

$$e^{c}_{\alpha}e^{d}_{\beta}e^{a}_{\alpha}e^{b}_{\beta}\frac{DS^{ab}}{Ds} + e^{c}_{\alpha}e^{d}_{\beta}S^{ab}\frac{D(e^{a}_{\alpha}e^{b}_{\beta})}{Ds} = e^{c}_{\alpha}e^{d}_{\beta}(P^{\alpha}U^{\beta} - P^{\beta}U^{\alpha}), \tag{5.12}$$

provided that

$$\frac{D(e^{a}_{\alpha}e^{b}_{\beta})}{Ds} = 0, \tag{5.13}$$

consequently,

$$\frac{DS^{cd}}{Ds} = (P^{c}U^{d} - P^{d}U^{c}). \tag{5.14}$$

Using the commutation relations as mentioned above, we obtain their corresponding deviation equations;

$$\frac{D^2\Phi^{c}}{Ds^2} = R^{c}{}_{d\beta\sigma}P^{d}U^{\beta}\Phi^{\sigma} + (R^{c}{}_{dab}S^{ab}U^{d})_{;\rho}\Phi^{\rho}, \tag{5.15}$$

and

$$\frac{D^2\Psi^{cd}}{Ds^2} = S^{l[d}\hat{R}^{c]}{}_{l\gamma\delta}U^{\gamma}\Psi^{\delta} + (P^{c}U^{d} - P^{d}U^{c})_{;\rho}\Psi^{\rho}. \tag{5.16}$$

5.3 Spinning Density and Spinning Density Deviation Equations for a class of Gauge theory

Let us define the following spin tensor

$$\bar{S}^{abc} = S^{bc}P^{a}, \tag{5.17}$$

Differentiating both sides using covariant derivative to get

$$\frac{D\bar{S}^{abc}}{Ds} = \frac{DS^{bc}}{Ds}P^{a} + \frac{DP^{a}}{Ds}S^{bc}. \tag{5.18}$$

Substituting (5.8) and (5.14) in (5.18) to obtain

$$\frac{D\bar{S}^{abc}}{Ds} = \frac{1}{2}R^{a}{}_{bcd}S^{bcd}. \tag{5.19}$$

We suggest the equivalent Bazanski Lagrangian

$$L = e^i_\rho e^j_\mu e^k_\nu S_{\rho\mu\nu} \frac{D\bar{\Psi}^{\rho\mu\nu}}{Ds} + f_{abc}\Psi^{abc}. \tag{5.20}$$

To become after taking the variation with respect to its corresponding deviation tensor $\Psi^{\rho\mu\nu}$

$$\frac{DS^{abc}}{Ds} = f^{abc}. \tag{5.21}$$

Similarly their corresponding geodesic deviation equation may be found as follows;

$$\frac{D^2\Psi^{abc}}{Ds^2} = S^{d[bc}R^{a]}{}_{dij}S^{dij} + f^{abc}{}_{;\gamma}\Psi^{\gamma}. \tag{5.22}$$

6 Discussion and Concluding Remarks

The spinning density tensor may play to express a spinning fluid. In this perspective the 3rd rank skew symmetric tensor turns to be specified to become $S^{\alpha\beta\gamma}$ skew symmetric in the last two indices in an approach to relate the spin density tensor with the Weyssenhoff tensor which is describing a spinning fluid. The problem of spinning density is vital to describe an extended object which can be expressible to plamsa fluid and heavy ion collision [17].

In the present work, we focused on spinning fluid tensor and its deviation one as a special case of applying spinning density tensor. In principle, a spinning fluid tensor is a special case of the spin density tensor with a skew-symmetry in the last two indices as expressed by Weyssenhoff tensor. Such a description has led many authors to geometrize it by means of using Riemannian-Cartan geometry, instead of the Riemannian geometry which gave a transition of GR to Einstein-Cartan theory of Gravity [10, 18]. This type of geometry has employed a new role of orthonormal frames to describe its internal composition. This gives rise to an insightful vision to examine the internal structure of a fluid element using a rotational potential vector $\Gamma^i_{j\mu}$ called spin connection able to describe torsion of space time beside the translational potential vector e^a_α. It is worth of mention Hehl et. al, were able to include spin tensor not only of energy momentum tensor $T_{\mu\nu}$ but also in angular momentum equations $\Omega^\mu_{\nu\rho}$ interms of torsion of space-time. This is clearly found Poincare gauge theory (PGT)[15, 19]; but if one introduces the absolute parallelism condition. [20, 21], one may relax the second equation of angular momentum which includes spin effects.

In our present work, we applied a tetrad formalism as expressed by means of orthonormal frames to express Spin density tensor in Equation (5.17). In this approach using the curvature tensor is defined by spin connection terms [16] . Using this type of amended curvature with symmetric affine connection, torsionless we obtain its corresponding the spin density tensor equation (5.19) and the spin deviation denity tensor equation (5.22) as expressed in mixed way anholonomic indices besides holonomic ones. Such a tendency to relate the internal effects. However, we are going to revisit the problem of spin density tensor and their corresponding spin and spin deviation tensor in the presence of modified absolute parallelism geometry including spin connection as described as to be expressed within the context of non-linear connection defined in Finsler geometry [24] will be examined in our future work.

Nevertheless, a spinning density tensor may be described in Clifford spaces [22, 23] in the following way such that $\hat{U}^{\mu} = \frac{\partial s}{\partial x^{\mu}}$ $\hat{S^{\mu\nu}} = \frac{\partial s}{\partial x^{\alpha\beta}}$ and $\hat{S}^{\mu\nu\rho} = \frac{\partial s}{\partial x^{\alpha\beta\gamma}}$ in terms of holograhic coordinates such that $s = s(x^{\mu}, x^{\mu\nu}, x^{\mu\nu\rho}, ...)$. However, from this perspective, replacing vectors to poly-vectors leading express torsion and curvature tensors viable to express the internal structure of spinning density tensor in Riemannian geometry. This issue will be studied in our future work as well.

References

[1] A. Papapetrou, "*Spinning test-particles in general relativity. I*," Proc. R. Soc. Lond. A, **209**, 248–258, (1951).

[2] M. E. Kahil, "*The spinning equations of motion for objects in AP-geometry*," ADAP, **3**, 136, (2018).

[3] S. L. Bazanski, "*Hamilton–Jacobi formalism for geodesics and geodesic deviations*," J. math. phys., **30**, 1018–1029, (1989).

[4] Ph. B. Yasskin and W. R. Stoeger "*Propagation equations for test bodies with spin and rotation in theories of gravity with torsion*," Phys. Rev. D, **21**, 2081–2094, (1980).

[5] K. Kleidis and N. K. Spyrou "*Geodesic motions versus hydrodynamic flows in a gravitating perfect fluid: Dynamical equivalence and consequences*," Class.Quant.Grav., **17**, 2965–2982, (2000).

[6] Z. Cao, K. Hattori, M. Hongo, Xu-Guang Huang and H. Tya "*Gyrohydrodynamics: Relativistic spinful fluid with strong vorticity*," Prog. Theor. Exp. Phys., arXiv:2205.08015.

[7] Th. Chrohok, H. Hermann and G. Rückner "*Spinning Fluids in Relativisitic Hydrodynamics*," Thchnicshe Meckanik, **22**, 1, (2002).

[8] F. Becattanini, W. Florkowski and E. Speranza, "*Spin tensor and its role in non-equilibrium thermodynamics*," Phys. Lett. B, **789**, 419–425, (2019).

[9] M. Mosheni, "*Spinning Fluid Cosmology*," Phys. Lett. B, **663**, 165, (2008).

[10] J.R. Ray and L. L. Smalley, "*Spinning fluids in general relativity*," Phys. Rev. D, **26**, 2619, (1982).

[11] M. E. Kahil, "*Spinning and Spinning Deviation Equations for Special Types of Gauge Theories of Gravity*," Gravit. Cosmol., **24**, 84–91, (2018).

[12] M. E. Kahil, "*Motion in Kaluza-Klein type theories*," J. math. phys., **47**, 052501, (2006).

[13] M. E. Kahil, "*Stability of Stellar System Orbiting SGR A**," Odessa Astro. Pub., **28**, 126, (2015).

[14] F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester "*General Relativity with Spin and Torsion: Foundations and Prospects*," Rev. Mod. Phys., **48**, 393–416, (1976).

[15] R. T. Hammond, "*Torsion gravity*," Rept. Prog. Phys., **65**, 599–649, (2002).

[16] P. Collins, A. Martin and E. Squires "*Particle Physics and Cosmology*," John Wiley and Sons, New York, (1989).

[17] A. D. Gallegos, Gürsory and A. Yarom "*Hydrodynamics, spin currents and torsion*," arXiv:2203.05044, (2022).

[18] L. L. Smalley and J. P. Krisch, "*String fluid dynamics in general relativity*," Class. Quant. Grav., **14**, ?, (1997).

[19] F. W. Hehl, "*Proceedings of the 6th Course of the International School of Cosmology and Gravitation on "Spin, Torsion and Supergravity*," P.G.Bergamann and V. de Sabatta, held at Erice, **1**, (1979).

[20] M. I. Wanas, "*Absolute parallelism geometry: Developments, Applications and Problems*," Studii şi Cercetări Ştiinţifice: Mat. Univ. Bacău, **10**, 297, (2001).

[21] M. I. Wanas, "*An Ap-Structure with Finslerian Flavor: I*," Mod. Phys. Lett. A, **24**, 1749–1762, (2009).

[22] M. E. Kahil, "*Motion in Clifford Space*," J. Mod. Phys., **11**, 1865–1873, (2020).

[23] C. Castro and M. Pavsic "*The Extended Relativity Theory in Clifford Spaces*," Prog. Phys., **1**, 31, (2005).

[24] M. E. Kahil, "*Spinning Equations for Objects of some Classes in Finslerian Geometry*," Gravit. Cosmol., **26**, 241, (2020).