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Current records and record range with some applications

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ABSTRACT

In a sequence of independent and identically distributed (iid) random variables, the upper (lower) current records and record range are studied. We derive general recurrence relations between the single and product moments for the upper and lower current records based on Weibull and positive Weibull distributions, as well as Pareto and negative Pareto distributions, respectively. Moreover, some asymptotic results for general current records are established. In addition, a recurrence relation and an explicit formula for the moments of record range based on the exponential distribution are given. Finally, numerical examples are presented to illustrate and corroborate theoretical results.

1. Introduction

Let $\{X_j\}$ be a sequence of independent and identically random variables (rv's) each distributed according to an absolutely continuous cumulative distribution function (cdf) $F_X(x) = P(X \leq x)$ and a probability density function (pdf) $f(x)$. An observation X_j will be called an upper record value if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. In a number of situations, one may store the largest and the smallest X values observed at the times when a new record of either kind (upper or lower) occurs. In this case these records are called current records. Specifically, we call U_n^c and L_n^c the nth upper and lower current records of the sequence X_n, respectively, when the nth record of any kind (either an upper or lower) is observed. It can be noticed that $U_{n+1}^c = U_n^c$ if $L_{n+1}^c < L_n^c$ and that $L_{n+1}^c = L_n^c$ if $U_{n+1}^c > U_n^c$, for all $n = 1, 2, \ldots,$ where by definition, $L_0^c = U_0^c = X_1$. The record range is then defined by $R_n^c = U_n^c - L_n^c$. The record range may also be defined as the nth record range in the sequence of the usual sample range $R_n = \max(X_1, X_2, \ldots, X_n) - \min(X_1, X_2, \ldots, X_n)$, where by definition $R_0^c = 0$ and $R_1^c = R_2$. Note that a new record range is attained once a new upper or lower record is observed.

Record values, as well as current records and record ranges, come up in several life situations. One application is industrial stress and life testing where measurements are taken sequentially and only observations that exceed, or only those that fall below, the current extreme value are recorded. It is interesting to note that there are situations in which only records are observed. Moreover, there are some situations wherein upper and lower records are observed together such as in the case of weather data. In these cases, it would therefore be interesting to consider current records. The current records can be used

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in general sequential method for model choice and outlier detection involving the record range (Basak, 2000). Actually, both current record values and record range can be detected in several real life situations. For example, the consistency of the production process is required to meet a product's specifications. If the current record range is large, then it is likely that large number of products will lie outside the specifications of the product. Ahmadi and Balakrishnan (2004) obtained distribution-free confidence intervals for quantiles based on current records and record range. Ahmadi and Balakrishnan (2005) established some reliability relationships between the original variable and the corresponding current records. Moreover, Ahmadi and Balakrishnan (2011) considered the case of records and order statistics jointly and discussed the construction of prediction intervals for order statistics. They observed that in the process of obtaining the ordinary record values, one usually observes the current records, and so it is worthwhile to use them in the construction of prediction intervals for order statistics. For more details about current records, record range and their applications, one may refer to Ahmadi and Balakrishnan (2008), Ahmadi, Razmkhaha, and Balakrishnan (2009) and Raqab (2009).

Houchens (1984) used an inductive argument to derive the pdf of U_n^c, L_n^c and R_n^c, based on an arbitrary cdf F_X, (in the sequel we write $U_n^c \parallel X$, $L_n^c \parallel X$ and $R_n^c \parallel X$ to indicate that these statistics are based on the cdf F_X), respectively by

$$f_{U_n^c \parallel X}(x) = 2^n f_X(x) \left\{ 1 - \bar{F}_X(x) \sum_{k=0}^{n-1} \frac{[-\log \bar{F}_X(x)]^k}{k!} \right\}, \tag{1.1}$$

$$f_{L_n^c \parallel X}(x) = 2^n f_X(x) \left\{ 1 - F_X(x) \sum_{k=0}^{n-1} \frac{[-\log F_X(x)]^k}{k!} \right\} \tag{1.2}$$

and

$$f_{R_n^c \parallel X}(r) = \frac{2^n}{(n-1)!} \int_{-\infty}^{\infty} f_X(r+x) f_X(x) \{-\log[1 - F_X(r+x) + F_X(x)]\}^{n-1} dx, \; 0 < r < \infty, \tag{1.3}$$

where $\bar{F}_X(x) = 1 - F_X(x)$.

Remark 1.1. It is worth mentioning that, the cdf of the nth current upper (lower) record is the mixture of the cdf of the $(n+1)$th current upper (lower) record and the cdf $\Gamma_{n+1}(-2\log \bar{F}(x))$ $(1 - \Gamma_{n+1}(-2\log F(x)))$, where $\Gamma_n(\theta) = \frac{1}{\Gamma(n)} \int_0^\theta t^{n-1} e^{-1} dt$ is the incomplete gamma function and the mixture constant is 1/2. This fact can be easily derived from the following two relations, which in turn can be easily proved from (1.1) and (1.2), respectively, by routine calculations.

$$F_{U_{n+1}^c \parallel X}(x) = 2F_{U_n^c \parallel X}(x) - \Gamma_{n+1}(-2\log \bar{F}(x))$$

and

$$F_{L_{n+1}^c \parallel X}(x) = 2F_{L_n^c \parallel X}(x) - (1 - \Gamma_{n+1}(-2\log \bar{F}(x))).$$

Since, we have

$$\begin{aligned} f_{-U_n^c \parallel X}(x) &= f_{U_n^c \parallel X}(-x) = 2^n f_X(-x) \left[1 - \bar{F}_X(-x) \sum_{k=0}^{n-1} \frac{(-\log \bar{F}_X(-x))^k}{k!} \right] \\ &= 2^n f_{-X}(x) \left[1 - \bar{F}_{-X}(x) \sum_{k=0}^{n-1} \frac{(-\log \bar{F}_{-X}(x))^k}{k!} \right], \end{aligned}$$

then, we get

$$-U_n^c \parallel X \overset{d}{=} L_n^c \parallel -X, \tag{1.4}$$

where "$X \overset{d}{=} Y$" means that the rv's X and Y have the same cdf's. The relation (1.4) yields that $U_n^c \parallel X \overset{d}{=} -L_n^c \parallel X$, if X is symmetric, i.e. $f_X(x) = f_X(-x) = f_{-X}(x)$.

Houchens (1984) deduced an outstanding representation of $U_n^c \parallel X$, when X has a negative exponential with parameter 2, i.e., $X \sim \mathrm{EX}(2)$. Namely,

$$U_n^c \parallel X \overset{d}{=} Y_0 + Y_1 + \cdots + Y_n, \tag{1.5}$$

where Y_i's are independent rv's such that $Y_0 \sim \mathrm{EX}(2)$ and the remaining $Y_i \sim \mathrm{EX}(1)$. An analogous representation for the lower current record can be easily obtained by applying (1.4). Namely, from (1.5), we have

$$-U_n^c \parallel X \overset{d}{=} -Y_0 - Y_1 - \cdots - Y_n \overset{d}{=} Z_0 + Z_1 + \cdots + Z_n,$$

where $Z_0 \sim \mathrm{EX}^+(2), Z_i \sim \mathrm{EX}^+(1),\ i = 1, 2, \ldots, n$, and $\mathrm{EX}^+(\beta)$ is the positive exponential cdf with parameter β. Thus, by applying again (1.5) and noting that $X \sim \mathrm{EX}(\beta) \Rightarrow -X \sim \mathrm{EX}^+(\beta)$, we get

$$L_n^c \parallel X \overset{d}{=} Z_0 + Z_1 + \cdots + Z_n, \tag{1.6}$$

where $X \sim \mathrm{EX}^+(2), Z_0 \sim \mathrm{EX}^+(2)$ and $Z_i \sim \mathrm{EX}^+(1),\ i = 1, 2, \ldots, n$.

In this paper, some recurrence relations on current records for Weibull and Pareto distributions are derived. Moreover, a recurrence relation on the moments of the record range under the exponential assumption is obtained. These results, in addition to providing some simple checks to test the accuracy of the computation of moments of current records and record range, can reduce the amount of direct computation of these moments, which is highly desirable since we have to determine these moments by numerical procedures. Moreover, some of these recurrence relations such as (2.7), (2.8) and (4.8) provide us with efficient prediction estimates for the moments of the future values of the current records and record range. In Section 2 we use the relation (1.5) to derive recurrence relations between the single and product moments for the upper record $U_n^c \parallel X$, in two cases. The first case is $X \sim \mathrm{W}(a, b)$, where $\mathrm{W}(a, b)$ is a Weibull cdf with parameters $a, b > 0$, i.e., $F_X(x) = 1 - \exp(-ax^b),\ x \geq 0$. The second case is $X \sim \mathrm{Par}(a)$, where $\mathrm{Par}(a)$ is a Pareto cdf with parameter $a > 0$, i.e., $F_X(x) = 1 - x^{-a},\ x \geq 1$. By using the relation (1.6), analogous results will be obtained for the lower current $L_n^c \parallel X$, where $X \sim \mathrm{W}^+(a, b)$, $\mathrm{W}^+(a, b)$ is a positive Weibull cdf with parameters $a, b > 0$, i.e., $F_X(x) = \exp(-a(-x)^b),\ x \leq 0$, and $X \sim \mathrm{Par}^-(a)$, $\mathrm{Par}^-(a)$ is the negative Pareto cdf with parameter $a > 0$, i.e., $F_X(x) = (-x)^{-a},\ x \leq -1$. In Section 3 the relations (1.5) and (1.6) are used to deduce some asymptotic results concerning the current records. Moreover, in Section 4 we derive an explicit formula for the moments of $R_n^c \parallel X$, when $X \sim \mathrm{EX}(\beta)$. In addition, a recurrence relation between the moments of $R_n^c \parallel X$ is derived. This recurrence relation, as well as the recurrence relations concerning the Pareto cdf, can be used to predict the moments of the future values of the record range and the upper and lower records. Finally, in Section 5 some numerical results are given to illustrate and corroborate theoretical results.

2. Recurrence relations for upper and lower current records

In this section, for any integer values m, k_1, k_2 we adopt the notations $\alpha_n^{(m)} = \mathrm{E}(U_n^c \parallel X)^m, \alpha_{n,n+1}^{(k_1,k_2)} = \mathrm{E}((U_n^c \parallel X)^{k_1}(U_{n+1}^c \parallel X)^{k_2}), X \sim \mathrm{W}(a, b)$; $\tilde{\alpha}_n^{(m)} = \mathrm{E}(U_n^c \parallel X)^m$, $\tilde{\alpha}_{n,n+1}^{(k_1,k_2)} = \mathrm{E}((U_n^c \parallel X)^{k_1}(U_{n+1}^c \parallel X)^{k_2}), X \sim \mathrm{EX}(2)$; $\beta_n^{(m)} = \mathrm{E}(L_n^c \parallel X)^m$, $\beta_{n,n+1}^{(k_1,k_2)} = \mathrm{E}((L_n^c \parallel X)^{k_1}(L_{n+1}^c \parallel X)^{k_2}), X \sim \mathrm{W}^+(a, b)$ and $\tilde{\beta}_n^{(m)} = \mathrm{E}(L_n^c \parallel X)^m$, $\tilde{\beta}_{n,n+1}^{(k_1,k_2)} = \mathrm{E}((L_n^c \parallel X)^{k_1}(L_{n+1}^c \parallel X)^{k_2}), X \sim \mathrm{EX}^+(2)$. Finally, $p_n^{(m)} = \mathrm{E}(U_n^c \parallel X)^m$, $p_{n,n+1}^{(k_1,k_2)} = \mathrm{E}((U_n^c \parallel X)^{k_1}(U_{n+1}^c \parallel X)^{k_2}), X \sim \mathrm{Par}(a)$, and $q_n^{(m)} = \mathrm{E}(L_n^c \parallel X)^m$, $q_{n,n+1}^{(k_1,k_2)} = \mathrm{E}((L_n^c \parallel X)^{k_1}(L_{n+1}^c \parallel X)^{k_2}), X \sim \mathrm{Par}^-(a)$. Theorems 2.1–2.4 give recurrence relations for $\alpha_n^{(m)}, \alpha_{n,n+1}^{(k_1,k_2)}, \beta_n^{(m)}, \beta_{n,n+1}^{(k_1,k_2)}, p_n^{(m)}, p_{n,n+1}^{(k_1,k_2)}, q_n^{(m)}$ and $q_{n,n+1}^{(k_1,k_2)}$.

Theorem 2.1. *For any integer values m, k_1, k_2, we get*

$$\alpha_{n+1}^{(mb)} = \sum_{j=0}^{m} \frac{m!}{j!(2a)^{m-j}} \alpha_n^{(jb)} \tag{2.1}$$

and

$$\alpha_{n,n+1}^{(k_1 b,k_2 b)} = \sum_{j=0}^{k_2} \frac{k_2!}{j!(2a)^{k_2-j}} \alpha_n^{((k_1+j)b)}. \tag{2.2}$$

Proof. First, we note that if $U_n^c \parallel Z$, where $Z \sim \mathrm{EX}(2)$, then the relation (1.5) enables us to write $(U_{n+1}^c \parallel Z)^m \overset{d}{=} (Y_0 + Y_1 + \cdots + Y_{n+1})^m \overset{d}{=} ((U_n^c \parallel Z) + Y_{n+1})^m \overset{d}{=} \sum_{j=1}^{m} \binom{m}{j} (U_n^c \parallel Z)^j Y_{n+1}^{m-j}$. In view of the independence of $U_n^c \parallel Z$ and Y_{n+1} (remember that all Y_i's are independent) and $Y_{n+1} \sim \mathrm{EX}(1)$, we get

$$\tilde{\alpha}_{n+1}^{(m)} = \sum_{j=0}^{m} \binom{m}{j} \tilde{\alpha}_n^{(j)} \mathrm{E}(Y_{n+1}^{m-j}) = \sum_{j=0}^{m} \frac{m!}{j!} \tilde{\alpha}_n^{(j)}. \tag{2.3}$$

Now, put $V_n = 2a(U_n^c \parallel X)^b$, where $X \sim \mathrm{W}(a, b)$. By using the fact that $Z = 2aX^b \sim \mathrm{EX}(2)$, the relation (1.1) yields

$$\begin{aligned} f_{V_n}(x) &= 2^n \left[\frac{(2a)^{-\frac{1}{b}}}{b} x^{\frac{1}{b}-1} f_X\left(\left(\frac{x}{2a}\right)^{\frac{1}{b}}\right) \right] \left[1 - \bar{F}_X\left(\left(\frac{x}{2a}\right)^{\frac{1}{b}}\right) \sum_{j=0}^{n-1} \frac{\left(-\log \bar{F}_X\left(\left(\frac{x}{2a}\right)^{\frac{1}{b}}\right)\right)^j}{j!} \right] \\ &= 2^n f_Z(x) \left[1 - \bar{F}_Z(x) \sum_{j=0}^{n-1} \frac{\left(-\log \bar{F}_Z(x)\right)^j}{j!} \right]. \end{aligned}$$

This means that $V_n \overset{d}{=} 2a(U_n^c \parallel X)^b \overset{d}{=} U_n^c \parallel Z$, where $Z \sim \mathrm{EX}(2)$. Thus,

$$\tilde{\alpha}_n^{(j)} = \mathrm{E}(U_n^c \parallel Z)^j = \mathrm{E}(V_n)^j = (2a)^j \mathrm{E}(U_n^c \parallel X)^{jb} = (2a)^j \alpha_n^{(jb)}.$$

Therefore, (2.3) yields

$$\alpha_{n+1}^{(mb)} = \frac{1}{(2a)^m}\tilde{\alpha}_{n+1}^{(m)} = \frac{1}{(2a)^m}\sum_{j=0}^{m}\frac{m!}{j!}\tilde{\alpha}_n^{(j)}$$
$$= \frac{1}{(2a)^m}\sum_{j=0}^{m}\frac{m!}{j!}(2a)^j\alpha_n^{(jb)} = \sum_{j=0}^{m}\frac{m!}{j!(2a)^{m-j}}\alpha_n^{(jb)}.$$

We now turn to prove the relation (2.2). First, we note that

$$(U_n^c \parallel Z)^{k_1}(U_{n+1}^c \parallel Z)^{k_2} \overset{d}{=} (U_n^c \parallel Z)^{k_1}((U_n^c \parallel Z) + Y_{n+1})^{k_2}$$
$$\overset{d}{=} (U_n^c \parallel Z)^{k_1}\sum_{j=0}^{k_2}\binom{k_2}{j}(U_n^c \parallel Z)^j (Y_{n+1})^{k_2-j},$$

which clearly yields (since all Y_i's are independent and $Y_{n+1} \sim \mathrm{EX}(1)$)

$$\tilde{\alpha}_{n,n+1}^{(k_1,k_2)} = \sum_{j=0}^{k_2}\frac{k_2!}{j!}\tilde{\alpha}_n^{(k_1+j)}. \tag{2.4}$$

Now, clearly $V_n^{k_1}V_{n+1}^{k_2} \overset{d}{=} (U_n^c \parallel Z)^{k_1}(U_{n+1}^c \parallel Z)^{k_2} \overset{d}{=} \left(2a(U_n^c \parallel X)^b\right)^{k_1}\left(2a(U_{n+1}^c \parallel X)^b\right)^{k_2}$, where $Z \sim \mathrm{EX}(2)$ and $X \sim \mathrm{W}(a,b)$. Thus by applying (2.4), (2.2) follows immediately. □

Theorem 2.2. *For any integer values m, k_1, k_2, we get*

$$\beta_{n+1}^{(mb)} = \sum_{j=0}^{m}\frac{m!(-1)^{(m-j)b}}{j!(2a)^{m-j}}\beta_n^{(jb)} \tag{2.5}$$

and

$$\beta_{n,n+1}^{(k_1b,k_2b)} = \sum_{j=0}^{k_2}\frac{k_2!(-1)^{(k_2-j)b}}{j!(2a)^{k_2-j}}\beta_n^{((k_1+j)b)}. \tag{2.6}$$

Proof. The proof is very similar to the proof of Theorem 2.1, with only the obvious changes. Indeed, we note that if $L_n^c \parallel Z$, where $Z \sim \mathrm{EX}^+(2)$, then the relation (1.6) yields

$$\tilde{\beta}_{n+1}^{(m)} = \sum_{j=0}^{m}\binom{m}{j}\tilde{\beta}_n^{(j)}\mathrm{E}(Z_{n+1}^{m-j}) = \sum_{j=0}^{m}\frac{m!(-1)^{m-j}}{j!}\tilde{\beta}_n^{(j)}$$

and

$$\tilde{\beta}_{n,n+1}^{(k_1,k_2)} = \sum_{j=0}^{k_2}\frac{k_2!(-1)^{(k_2-j)b}}{j!}\tilde{\beta}_n^{(k_1+j)}.$$

The remaining part of the proof is now followed by using the fact that, if $X \sim \mathrm{W}^+(a,b)$, then $Z = -2a(-X)^b \sim \mathrm{EX}^+(2)$. □

Theorem 2.3. *For any integer values m, k_1, k_2, we get*

$$p_{n+1}^{(m)} = \frac{2a}{2a-m}p_n^{(m)}, \quad \forall m < 2a, \tag{2.7}$$

and

$$p_{n,n+1}^{(k_1,k_2)} = \frac{2a}{2a-k_2}p_n^{(k_1+k_2)}, \quad \forall k_2 < 2a. \tag{2.8}$$

Proof. By using the fact that $X = \exp\left(\frac{Y}{2a}\right) \sim \mathrm{Par}(a)$, if $Y \sim \mathrm{EX}(2)$, we get $(U^c_{n+1} \parallel X)^m \overset{d}{=} \exp\left(\frac{m}{2a}(U^c_{n+1} \parallel Y)\right)$. By taking the expectation we get

$$p^{(m)}_{n+1} = M_{n+1}\left(\frac{m}{2a}\right), \tag{2.9}$$

where $M_n(t) = \mathrm{E}\Big(\exp(t(U^c_n \parallel Y))\Big)$ is the moment generating function of $U^c_n \parallel Y$. On the other hand, in view of (1.5), we get

$$M_{n+1}(t) = \mathrm{E}\Big(\exp(t(Y_0 + Y_1 + \cdots + Y_n + Y_{n+1}))\Big) = M_n(t)\mathrm{E}\Big(e^{tY_{n+1}}\Big) = M_n(t)(1-t)^{-1}, \quad t < 1. \tag{2.10}$$

Thus, by combining (2.9) and (2.10) we get the relation (2.7). To prove the relation (2.8), we start with the obvious relation

$$(U^c_n \parallel X)^{k_1}(U^c_{n+1} \parallel X)^{k_2} \overset{d}{=} \exp\left(\frac{k_1 + k_2}{2a}(Y_0 + Y_1 + \cdots + Y_n)\right)\exp\left(\frac{k_2}{2a}Y_{n+1}\right),$$

which yields (2.8), upon taking the expectation. The theorem is proved. □

Theorem 2.4. *For any integer values* m, k_1, k_2, *we get*

$$q^{(m)}_{n+1} = \frac{2a}{m-2a}q^{(m)}_n, \quad \forall m < 2a, \quad \textit{and} \quad q^{(k_1,k_2)}_{n,n+1} = \frac{2a}{2a-k_2}q^{(k_1+k_2)}_n, \quad \forall k_2 < 2a.$$

Proof. By using the fact that $X = -\exp\left(-\frac{Y}{2a}\right) \sim \mathrm{Par}^-(a)$, if $Y \sim \mathrm{EX}^+(2)$, we can easily show that the proof is very similar to the proof of Theorem 2.3, with only the obvious changes. □

3. Some asymptotic results

In this section, we adopt the abbreviation $(\xrightarrow[n]{})$ for the convergence, as $n \to \infty$. Moreover, $F_{X_n} \xrightarrow[n]{w} F_X$ (or equivalently $X_n \xrightarrow[n]{d} X$) stands for the weak convergence of the sequence of cdf $\{F_{X_n}\}$ to the limit cdf $\{F_X\}$ (or equivalently the sequence of rv's $\{X_n\}$ converges to the limit rv X in the distribution).

The relations (1.5) and (1.6) open the way for several asymptotic results for $U^c_n \parallel X$ and $L^c_n \parallel X$, respectively, where X has an arbitrary continuous cdf F_X. In this section we will give some asymptotic results concerning the two current records $U^c_n \parallel X$ and $L^c_n \parallel X$. The detailed discussion will be given for the upper current records. By defining the function $U_X(x) = -\log(1 - F_X(x))$, we clearly get $U_X(X) \sim \mathrm{EX}(1)$. Therefore, $T_n = 2U_X(U^c_n \parallel X) \overset{d}{=} U^c_n \parallel Y$, where $Y \sim \mathrm{EX}(2)$, which, in view of (1.5), implies $T_n \overset{d}{=} Y_0 + Y_1 + \cdots + Y_n$, where Y_i's are independent rv's such that $Y_0 \sim \mathrm{EX}(2)$ and the remaining $Y_i \sim \mathrm{EX}(1)$. We have thus the following essential lemma.

Lemma 3.1. *We have*

$$\frac{U_X(U^c_n \parallel X) - \frac{n}{2}}{\frac{\sqrt{n}}{2}} \xrightarrow[n]{d} Z \sim \mathrm{N}(0, 1), \tag{3.1}$$

where $\mathrm{N}(0, 1)$ *stands for the standard normal distribution.*

Proof. In view of the central limit theorem, we get $\frac{S_n - n}{\sqrt{n}} \xrightarrow[n]{d} Z$, where $T_n - Y_0 \overset{d}{=} S_n = \sum_{i=1}^n Y_i$. On the other hand, we have $\frac{S_n - n}{\sqrt{n}} \overset{d}{=} \frac{T_n - n}{\sqrt{n}} - \frac{Y_0}{\sqrt{n}}$ and $P\left(\frac{Y_0}{\sqrt{n}} \le x\right) = (1 - e^{-x\sqrt{n}})\mathrm{I}_{[0,\infty)}(x) \xrightarrow[n]{} \mathrm{I}_{[0,\infty)}(x)$, where $I_A(x) = 0, 1$ if $x \notin A, x \in A$, respectively. Thus, by applying Lemma 2.2.1 of Galambos (1987), we get $\frac{T_n - n}{\sqrt{n}} \xrightarrow[n]{d} Z$, which implies the claimed result. □

Lemma 3.1 leads to the following additional general result.

Lemma 3.2. *There are constants* $a_n > 0$ *and* b_n, *for which* $P(U^c_n \parallel X \le a_n x + b_n) \xrightarrow[n]{w} T(x)$, *where* $T(x)$ *is nondegenerate cdf, if and only if*

$$\frac{U_X(a_n x + b_n) - n}{\sqrt{n}} \xrightarrow[n]{} g(x), \tag{3.2}$$

where g is finite on an interval and has at least two growth points. Moreover, $T(x) = \mathrm{N}(g(x))$, *where* $\mathrm{N}(x)$ *is the standard normal cdf.*

Proof. Since the function $U_X(x)$ is monotone nondecreasing, the proof follows from the following equalities

$$\{U_n^c \parallel X \le a_n x + b_n\} = \{U_X(U_n^c \parallel X) \le U_X(a_n x + b_n)\}$$
$$= \left\{ \frac{U_X(U_n^c \parallel X) - \frac{n}{2}}{\frac{\sqrt{n}}{2}} \le \frac{U_X(a_n x + b_n) - \frac{n}{2}}{\frac{\sqrt{n}}{2}} \right\}$$

and by applying Lemma 3.1. □

Lemmas 3.1 and 3.2 enable us to deduce the following theorem, which fully characterizes the class of possible limit laws of the upper current records.

Theorem 3.1. *Let* $a_n > 0$ *and* b_n *be suitable constants for which* (3.2) *is satisfied, where* $g(x)$ *has at least two points of increase, i.e.,* $T(x) = \mathrm{N}(g(x))$ *is a nondegenerate cdf. Then for every real value* τ, *the function* $g(x)$ *satisfies the relation*

$$g(a(\tau)x + b(\tau)) = g(x) - \tau, \tag{3.3}$$

where

$$\frac{a_{[n-\tau\sqrt{n}]}}{a_n} \xrightarrow[n]{} a(\tau), \qquad \frac{b_{[n-\tau\sqrt{n}]} - b_n}{a_n} \xrightarrow[n]{} b(\tau) \tag{3.4}$$

and $[x]$ *is the integer part of* x. *Moreover, there are only three possible types of the function* $g(x)$. *Namely*

$$g_1(x,\alpha) = \begin{cases} -\infty, & x \le 0, \\ \alpha \log x, & x > 0, \end{cases} \qquad g_2(x,\alpha) = \begin{cases} -\alpha \log |x|, & x \le 0, \\ \infty, & x > 0, \end{cases} \quad \text{and} \quad g_3(x) = x, \quad \forall\, x. \tag{3.5}$$

Proof. For any real τ, let $s = s(n) = [n - \tau\sqrt{n}]$. Then, we can write

$$\mathcal{U}_s(a_s x + b_s) = \sqrt{\frac{n}{s}}\, \mathcal{U}_n(a_s x + b_s) + \frac{n-s}{\sqrt{s}},$$

where $\mathcal{U}_n(x) = \frac{U_X(x) - \frac{n}{2}}{\frac{\sqrt{n}}{2}}$. Moreover, it is easy to verify that $\sqrt{\frac{n}{s}} \xrightarrow[n]{} 1$ and $\frac{n-s}{\sqrt{s}} \sim \tau\sqrt{\frac{n}{s}} \xrightarrow[n]{} \tau$. Consequently, Lemma 3.2 yields $P(U_n^c \parallel X \le a_n x + b_n) \xrightarrow[n]{w} \mathrm{N}(g(x) - \tau)$, where $\mathrm{N}(g(x) - \tau)$ is a nondegenerate cdf. On the other hand, we have the relation $P(U_n^c \parallel X \le a_n x + b_n) \xrightarrow[n]{w} \mathrm{N}(g(x))$. Thus an application of Khintchine's convergence theorem implies the existence of two real constants $a(\tau) > 0$ and $b(\tau)$ which satisfy (3.4) and $\mathrm{N}(g(a(\tau)x + b(\tau))) = \mathrm{N}(g(x) - \tau)$. The last relation implies (3.3). Now, in (1.5), let $g^\star(x) = \exp(g(x))$, it follows that $Ag^\star(a(\tau)x + b(\tau)) = g^\star(x)$, where $A = \exp(\tau)$. Smirnov (1952) proved in Part II, Theorem 5, that the only monotonically increasing nonnegative solutions of this functional equation, within a linear transformation of the argument, can be

$$g_1^\star(x,\alpha) = \begin{cases} 0, & x \le 0, \\ x^\alpha, & x > 0, \end{cases} \qquad g_2^\star(x,\alpha) = \begin{cases} |x|^{-\alpha}, & x \le 0, \\ \infty, & x > 0, \end{cases} \quad \text{and} \quad g_3^\star(x) = e^x, \quad \forall x.$$

The corresponding functions $g(x)$ are $g_i(x,\alpha),\ i = 1, 2$, and $g_3(x)$, respectively. That proves the theorem. □

Remark 3.1. It is easy to verify that the class of possible limiting types of lower current records is the same as given in Theorem 3.1. Moreover, it is worth mentioning that the class of possible limiting types of the intermediate order statistics (Wu, 1966), the class of the possible limiting types of the usual records (Resnik, 1973 and Arnold, Balakrishnan, & Nagaraja, 1998) and the class of the possible limiting types of the current record (3.5) are the same. This resemblance is due to the fact that the three classes are governed by the same functional equation (3.3).

Corollary 3.1 (*The Continuation Property*). *If the sequence* $\{P(U_n^c \parallel X \le a_n x + b_n)\}$ *converges weakly on some compact interval* $I = [c, d]$ *to a continuous and strictly increasing function* T, *then* $\{P(U_n^c \parallel X \le a_n x + b_n)\}$ *converges weakly on the whole real line (to one of the three limits* $\mathrm{N}(g_i(x,\alpha)),\ i = 1, 2$, *and* $\mathrm{N}(g_3(x))$*).*

Proof. The proof follows by combining Remark 3.1 and Theorem 2.1 in Barakat (2010). □

Since the record range is the ordinary $(n-1)$th record value in a non-independent sequence of rv's, the study of the asymptotic behavior of the record range is considered a real challenge. The only case in which this can be easily done is when X is a uniform(0, 1). Although, in this case the record range itself cannot be normalized to obtain a non-degenerate limit, but we have the following interesting result.

Theorem 3.2. *Let X be a uniformly $(0,\ 1)$ distributed rv. Then*

$$F_{R_n^{\star c}\|X}(\sqrt{n}x+n)\xrightarrow[n]{w}\mathrm{N}(x), \tag{3.6}$$

where $R_n^{\star c}\ \|\ X=-2\log(1-R_n^c\ \|\ X)$*. Clearly, the relation* (3.6) *provides us with the asymptotic relation* $F_{R_n^c\|X}(r)\sim 1-\mathrm{N}(\sqrt{n}+\frac{2}{\sqrt{n}}\log(1-r)),\ \forall 0\le r\le 1$*, as* $n\to\infty$*.*

Proof. Directly from (1.3) we get

$$F_{R_n^c\|X}(r)=\frac{2}{\Gamma(n)}\int_0^r(1-v)\Big[-2\log(1-v)\Big]^{n-1}dv=\Gamma_n(-2\log(1-r)).$$

Thus, we get $F_{R_n^{\star c}\|X}(r)=\Gamma_n(r)$. The asymptotic relation (3.6) follows from the fact that $\Gamma_n(\sqrt{n}x+n)\xrightarrow[n]{}\mathrm{N}(x)$, uniformly for all x (cf. Johnson & Kotz, 1970). □

4. The moments of the record range based on the exponential cdf

In this section, our aim is to derive a recurrence relation between the single moments $\delta_n^{(m)}(\beta)=\mathrm{E}(R_n^c\ \|\ X)^m,\ n=2,3,4,\ldots$, where $X\sim\mathrm{EX}(\beta)$ and m is any positive integer. Moreover, we give the exact form of $\delta_n^{(m)}(\beta)$.

We will frequently use the general Hurwitz–Lerch Zeta function (cf. Jankov, 2011, Chapter 6)

$$Q(z,M,\ell)=\sum_{k=0}^{\infty}\frac{z^k}{(k+\ell)^M}=\frac{1}{\Gamma(M)}\int_0^\infty\frac{\theta^{M-1}e^{-\ell\theta}}{1-ze^{-\theta}}d\theta, \tag{4.1}$$

where $\ell=1,2,\ldots,\ M=2,3,4,\ldots,|z|\le 1$ and $\Gamma(\cdot)$ is the complete gamma function. As a special case, the Hurwitz-Zeta function, is defined by $\zeta(M,\ell)=Q(1,M,\ell)$. Moreover, the usual Riemann Zeta function is defined by $\zeta(M)=Q(1,M,1)$. Finally, the following lemma is essential for our next study.

Lemma 4.1. *We have the following three integrations*

$$\int_0^\infty\frac{\theta^{M-1}e^{-\theta}}{(1-e^{-\theta})}d\theta=\Gamma(M)\zeta(M,1),\quad M=1,2,\ldots, \tag{4.2}$$

$$\int_0^\infty\frac{\theta^{M-1}e^{-\theta}}{(1-e^{-\theta})^2}d\theta=\Gamma(M)\zeta(M-1,1),\quad M=2,3,\ldots, \tag{4.3}$$

$$\int_0^\infty\frac{\theta^{M-1}e^{-l\theta}}{(1-e^{-\theta})^2}d\theta=I(\ell,M),\quad M=2,3,\ldots, \tag{4.4}$$

where ℓ is positive integer and $I(\ell,M)=I(\ell-1,M)-\Gamma(M)\zeta(M,\ell-1)$*. As special cases:*

$$I(1,M)=\int_0^\infty\frac{\theta^{M-1}e^{-\theta}}{(1-e^{-\theta})^2}d\theta=\Gamma(M)\zeta(M-1,1), \tag{4.5}$$

$$I(2,M)=I(1,M)-\Gamma(M)\zeta(M,1)=\Gamma(M)(\zeta(M-1,1)-\zeta(M,1)), \tag{4.6}$$

$$I(3,M)=I(2,M)-\Gamma(M,2)=\Gamma(M)(\zeta(M-1,1)-\zeta(M,1)-\zeta(M,2)). \tag{4.7}$$

Proof. The integration (4.2) is obvious. The relation (4.3) follows by applying the integration by parts

$$\begin{aligned}\int_0^\infty\frac{\theta^{M-1}e^{-\theta}}{(1-e^{-\theta})^2}d\theta&=-\int_0^\infty\theta^{M-1}d\left(\frac{e^{-\theta}}{1-e^{-\theta}}\right)\\&=(M-1)\int_0^\infty\theta^{M-2}\frac{e^{-\theta}}{1-e^{-\theta}}d\theta=\Gamma(M)\zeta(M-1,1).\end{aligned}$$

To prove (4.4), we first note that $\frac{e^{-\ell\theta}}{(1-e^{-\theta})^2}+\frac{e^{-\theta(\ell-1)}}{(1-e^{-\theta})}=\frac{e^{-\theta(\ell-1)}}{(1-e^{-\theta})^2}$. Thus,

$$\begin{aligned}I(\ell,M)&=\int_0^\infty\frac{\theta^{M-1}e^{-\theta(\ell-1)}}{(1-e^{-\theta})^2}d\theta-\int_0^\infty\frac{\theta^{M-1}e^{-\theta(\ell-1)}}{(1-e^{-\theta})}d\theta\\&=I(\ell-1,M)-\zeta(M,\ell-1)\Gamma(M).\end{aligned}$$

The lemma is proved. □

Theorem 4.1. *For any positive integers $m \geq 1$ and $n \geq 2$, we get*

$$\begin{aligned}\delta_{n+1}^{(m)}(\beta) - 2\delta_n^{(m)}(\beta) &= \beta^m\Gamma(m+1)\Big(\zeta(m+1,1)+\zeta(m+1,2)\Big)\\ &\quad - \frac{2^n\beta^m\Gamma(n+m+1)}{\Gamma(n+1)}\Big[\zeta(n+m,1)-\zeta(n+m+1,1)+\zeta(n+m+1,2)\Big]\\ &\quad - \sum_{k=1}^{n-1}\frac{2^n\beta^m\Gamma(k+m+1)}{\Gamma(k+1)}\Big[\zeta(k+m,1)-\zeta(k+m+1,1)-\zeta(k+m+1,2)\Big]. \end{aligned} \tag{4.8}$$

Moreover, we have

$$\begin{aligned}\delta_n^{(m)}(\beta) &= (2^n-1)\beta^m\Gamma(m+1)\zeta(m+1,1) - \beta^m\Gamma(m+1)\zeta(m+1,2)\\ &\quad + \sum_{k=1}^{n-1}\frac{\beta^m\Gamma(k+m+1)}{\Gamma(k+1)}\Big[(2^n-2^k)\Big(\zeta(k+m+1,1)-\zeta(k+m,1)\Big) - 2^k\zeta(k+m+1,2)\Big]. \end{aligned} \tag{4.9}$$

Proof. If $X \sim \text{EX}(\beta)$, then (1.3) can be written in the form

$$f_{R_n^c\|X}(r) = \frac{2^n e^{-r/\beta}}{\beta^2(n-1)!}\int_0^\infty e^{-2x/\beta}[-\log(1-ae^{-x/\beta})]^{n-1}dx.$$

Upon using successively the transformations $z = e^{-x/\beta}$ and $\xi = -\log[1-az]$, where $a = 1-e^{-\frac{r}{\beta}} \geq 0$, we get

$$\begin{aligned} f_{R_n^c\|X}(r) &= \frac{2^n e^{r/\beta}}{\beta(n-1)!}\int_0^1 z[-\log(1+az)]^{n-1}dz = \frac{2^n e^{-r/\beta}}{\beta a^2(n-1)!}\int_0^{-\log(1-a)} e^{-\xi}(1-e^{-\xi})\xi^{n-1}d\xi\\ &= \frac{2^n e^{-r/\beta}}{\beta a^2(n-1)!}\int_0^{r/\beta} e^{-\xi}(1-e^{-\xi})\xi^{n-1}d\xi = \frac{e^{-r/\beta}}{a^2\beta}J_n(r), \end{aligned} \tag{4.10}$$

where $J_n(r) = \frac{2^n}{(n-1)!}\int_0^{r/\beta} e^{-\xi}(1-e^{-\xi})\xi^{n-1}d\xi$. Therefore,

$$J_{n+1}(r) = \frac{2^{n+1}}{n!}\int_0^{r/\beta} e^{-\xi}(1-e^{-\xi})\xi^n d\xi.$$

Upon integrating by parts (by differentiating ξ^n and integrating $(e^{-\xi}-e^{-2\xi})$), we get

$$\begin{aligned} J_{n+1}(r) &= \frac{2^{n+1}}{n!}\left[\frac{1}{2}\left(\frac{r}{\beta}\right)^n e^{-2r/\beta} - \left(\frac{r}{\beta}\right)^n e^{-r/\beta} + \frac{n}{2}\int_0^{r/\beta} e^{-\xi}(2-e^{-\xi})\xi^{n-1}d\xi\right]\\ &= -\frac{2^n r^n e^{-r/\beta}a}{n!\beta^n} - \frac{2^n r^n e^{-r/\beta}}{n!\beta^n} + J_n(r) + \frac{2^n}{(n-1)!}\int_0^{r/\beta} e^{-\xi}\xi^{n-1}d\xi\\ &= -\frac{2^n r^n e^{-r/\beta}a}{n!\beta^n} - \frac{2^n r^n e^{-r/\beta}}{n!\beta^n} + 2J_n(r) + \frac{2^n}{(n-1)!}\int_0^{r/\beta} e^{-2\xi}\xi^{n-1}d\xi. \end{aligned}$$

Thus, we have

$$f_{R_{n+1}^c\|X}(r) - 2f_{R_n^c\|X}(r) = -\frac{2^n r^n e^{-2r/\beta}a}{n!\beta^{n+1}a^2} - \frac{2^n r^n e^{-2r/\beta}}{n!\beta^{n+1}a^2} + \frac{2^n e^{-r/\beta}}{a^2\beta(n-1)!}\int_0^{r/\beta} e^{-2\xi}\xi^{n-1}d\xi.$$

By using the basic relation

$$\frac{1}{(n-1)!}\int_0^t \lambda^n x^{n-1}e^{-\lambda x}dx = 1 - \sum_{k=0}^{n-1}\frac{e^{-\lambda t}(\lambda t)^k}{k!}, \tag{4.11}$$

we get

$$f_{R_{n+1}^c\|X}(r) - 2f_{R_n^c\|X}(r) = A_1(r) + A_2(r) + A_3(r) + A_4(r), \tag{4.12}$$

where $A_1(r) = -\frac{2^n r^n e^{-2r/\beta}}{n!\beta^{n+1}a}$, $A_2(r) = -\frac{2^n r^n e^{-2r/\beta}}{n!\beta^{n+1}a^2}$, $A_3(r) = \frac{e^{-r/\beta}}{a\beta}(1+e^{-r/\beta})$ and $A_4(r) = -\frac{e^{-r/\beta}}{a^2\beta}\sum_{k=1}^{n-1}\frac{2^k e^{-2r/\beta} r^k}{\beta^k k!}$. Now, for any integers $m \geq 1$ and $n \geq 2$, by using the relations (4.1)–(4.7), we can easily calculate the following integrations.

$$\begin{aligned}
\int_0^\infty r^m A_1(r)dr &= \frac{-2^n}{\beta^{n+1}n!}\int_0^\infty \frac{r^{n+m}e^{-2r/\beta}}{(1-e^{-r/\beta})}dr \\
&= \frac{-2^n\beta^{n+m+1}}{n!\beta^{n+1}}\int_0^\infty \frac{\theta^{n+m}e^{-2\theta}}{(1-e^{-\theta})}d\theta \\
&= \frac{-2^n\beta^m}{n!}\zeta(n+m+1,2)\Gamma(n+m+1), \qquad (4.13)
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty r^m A_2(r)dr &= -\frac{2^n\beta^{n+m+1}}{n!\beta^{n+1}}\int_0^\infty \frac{\theta^{n+m}e^{-2\theta}}{(1-e^{-\theta})^2}d\theta = -\frac{2^n\beta^m}{n!}I(2,n+m+1) \\
&= -\frac{2^n\beta^m}{n!}\Gamma(n+m+1)(\zeta(m+n,1)-\zeta(m+n+1,1)), \qquad (4.14)
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty r^m A_3(r)dr &= \frac{1}{\beta}\left[\int_0^\infty \frac{r^m e^{-r/\beta}}{(1-e^{-r/\beta})}dr + \int_0^\infty \frac{r^m e^{-2r/\beta}}{(1-e^{-r/\beta})}dr\right] \\
&= \frac{\beta^{m+1}}{\beta}\left[\int_0^\infty \frac{\theta^m e^{-\theta}}{(1-e^{-\theta})}d\theta + \int_0^\infty \frac{\theta^m e^{-2\theta}}{(1-e^{-\theta})}d\theta\right] \\
&= \beta^m\Big[\zeta(m+1,1)+\zeta(m+1,2)\Big]\Gamma(m+1), \qquad (4.15)
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty r^m A_4(r)dr &= -\sum_{k=1}^{n-1}\frac{2^k}{\beta^{k+1}k!}\int_0^\infty \frac{r^{k+m}e^{-3r/\beta}}{(1-e^{-r/\beta})^2}dr \\
&= -\sum_{k=1}^{n-1}\frac{2^k\beta^{k+m+1}}{\beta^{k+1}\Gamma(k+1)}\int_0^\infty \frac{\theta^{k+m}e^{-3\theta}}{(1-e^{-\theta})^2}d\theta = -\sum_{k=1}^{n-1}\frac{2^k\beta^m}{\Gamma(k+1)}I(3,k+m+1) \\
&= -\sum_{k=1}^{n-1}\frac{2^k\beta^m\Gamma(m+k+1)}{\Gamma(k+1)}\Big(\zeta(k+m,1)-\zeta(k+m+1,1)-\zeta(k+m+1,2)\Big). \qquad (4.16)
\end{aligned}$$

Therefore, by noting that $\delta_n^{(m)}(\beta) = \int_0^\infty r^m f_{R_n^c\|X}(r)dr$ and by combining (4.12)–(4.16), we get the relation (4.8). We now turn to the proof of (4.9). By starting with the relation (4.10) and by using the relation (4.11), we get

$$\begin{aligned}
f_{R_n^c\|X}(r) &= \frac{2^n e^{-r/\beta}}{\beta a^2(n-1)!}\left(\int_0^{r/\beta} e^{-\xi}\xi^{n-1}d\xi - \int_0^{r/\beta} e^{-2\xi}\xi^{n-1}d\xi\right) \\
&= \frac{(2^n-1)e^{-r/\beta}}{\beta a^2} - \frac{2^n e^{-r/\beta}}{\beta a^2}\sum_{k=1}^{n-1}\frac{r^k e^{-r/\beta}}{\beta^k k!} + \frac{e^{-r/\beta}}{\beta a^2}\sum_{k=1}^{n-1}\frac{2^k r^k e^{-2r/\beta}}{\beta^k k!}.
\end{aligned}$$

Upon using the relations (4.1)–(4.7), and after some routine calculations we get

$$\begin{aligned}
\delta_n^{(m)}(\beta) &= (2^n-1)\beta^m\Gamma(m+1)\zeta(m,1) + \sum_{k=0}^{n-1}\frac{\beta^m\Gamma(k+m+1)}{\Gamma(k+1)}\Big[(2^n-2^k)\Big(\zeta(k+m+1,1) \\
&\quad -\zeta(k+m,1)\Big) - 2^k\zeta(k+m+1,2)\Big],
\end{aligned}$$

which directly leads to (4.9). The theorem is proved. □

The stopping time. For i.i.d. sample from the cdf $F_X(x)$ and pdf $f_X(x)$, let $\mathcal{N}$ denote the stopping time

$$\mathcal{N} = \inf\{n > 0 : R_n^c > c,\ c > 0\},$$

where c is fixed but arbitrary. Thus $\mathcal{N}$ simply gives the waiting time until the record range of an i.i.d. sample exceeds a given value c. Basak (2000) showed that, for any $0 < c < \infty$, we have

$$\mathrm{E}(\mathcal{N}) = \int_0^\infty \frac{2f_X(x)}{1-F_X(x+c)+F_X(x)}dx - 1$$

and

$$\mathrm{var}(\mathcal{N}) = 2 - \mathrm{E}(\mathcal{N}) - \mathrm{E}^2(\mathcal{N}) - 8\int_{-\infty}^\infty \frac{\log[1-F_X(x+c)+F_X(x)]}{1-F_X(x+c)+F_X(x)}f_X(x)dx.$$

Table 1
The mean $\delta_n^{(1)}(\beta)$ and the variance $\sigma_n^2(\beta) = \delta_n^{(2)}(\beta) - (\delta_n^{(1)}(\beta))^2$ of the record range.

n	$\delta_n^{(1)}(0.5)$	$\sigma_n^2(0.5)$	$\delta_n^{(1)}(1)$	$\sigma_n^2(1)$	$\delta_n^{(1)}(1.5)$	$\sigma_n^2(1.5)$
2	0.855	0.365	1.710	1.459	2.565	3.283
3	1.161	0.446	2.322	1.782	3.483	4.010
4	1.444	0.514	2.888	2.056	4.332	4.625
5	1.714	0.577	3.428	2.308	5.142	5.193
6	1.977	0.638	3.953	2.552	5.930	5.743
7	2.235	0.699	4.469	2.794	6.704	6.286
8	2.490	0.759	4.980	3.035	7.470	6.829
9	2.743	0.819	5.487	3.278	8.230	7.375
10	2.996	0.880	5.991	3.521	8.987	7.923

Therefore, if $X \sim \mathrm{EX}(\beta)$, we get in view (4.1)

$$\mathrm{E}(\mathcal{N}) = 2Q(\varsigma, 1, 1) - 1,$$

and

$$\mathrm{var}(\mathcal{N}) = 2 + 2Q(\varsigma, 1, 1) - 4Q^2(\varsigma, 1, 1) + \frac{4c^2}{\varsigma\beta^2},$$

where $\varsigma = 1 - e^{-c/\beta}$.

5. Numerical examples

Example 5.1. In Table 1, we apply the relation (4.9), by using Mathematica 8.0, to calculate the exact value of the mean $\delta_n^{(1)}(\beta)$ and the variance $\sigma_n^2(\beta) = \delta_n^{(2)}(\beta) - (\delta_n^{(1)}(\beta))^2$ for $n = 2, 3, \ldots, 10$, $\beta = 0.5, 1, 1.5$. We note that the mean and the variance values of the record range increase by increasing the value of the parameter β. Moreover, for all values $n = 2, 3, \ldots, 10$, and $\beta = 0.5, 1, 1.5$, we have $\delta_n^{(1)}(\beta) < \delta_{n+1}^{(1)}(\beta) < 2\delta_n^{(1)}(\beta)$ and $\sigma_n^2(\beta) < \sigma_{n+1}^2(\beta) < 2\sigma_n^2(\beta)$.

Example 5.2. We note that the recurrence relations in Theorems 2.3, 2.4 and 4.1 provide us with efficient prediction estimates for the moments of the future values of the current records and record range. For example, let $\bar{p}_n^{(m)}$ and $\bar{\delta}_n^{(m)}(\beta)$ be any unbiased and consistent estimates for $p_n^{(m)}$ and $\delta_n^{(m)}(\beta)$, respectively. Then, clearly the predictions $\hat{p}_{n+1}^{(m)} = \bar{p}_{n+1}^{(m)}$ and $\hat{\delta}_{n+1}^{(m)}(\beta) = \bar{\delta}_{n+1}^{(m)}(\beta)$, resulted from (2.7) and (4.8), respectively, are unbiased and consistent prediction estimates for $p_{n+1}^{(m)}$ and $\delta_{n+1}^{(m)}(\beta)$, respectively (note that $\mathrm{Var}(\hat{p}_{n+1}^{(m)}) = \left(\frac{2a}{2a-m}\right)^2 \mathrm{Var}(\bar{p}_n^{(m)})$ and $\mathrm{Var}(\hat{\delta}_{n+1}^{(m)}(\beta)) = 4\mathrm{Var}(\bar{\delta}_n^{(m)}(\beta))$). To illustrate numerically this interesting and useful fact, we simulate samples each of size 20 from the cdf of the upper record (1.1), for Pareto cdf for the values $n = 1, 2, 3, 4$; $a = 1.5, 2, 2.5$ (we use the recurrence relation defined in Remark 1.1 and Mathematica 8.0). By calculating the usual arithmetic mean for each of these samples, we get the unbiased and consistent estimates $\bar{p}_n^{(1)}$, $n = 1, 2, 3, 4$; $a = 1.5, 2, 2.5$, for the corresponding true means $p_n^{(1)}$, $n = 1, 2, 3, 4$; $a = 1.5, 2, 2.5$. By using these estimates, for $n = 1, 2, 3$; $a = 1.5, 2, 2.5$, and the recurrence relation (2.7), we get the corresponding predictions $\hat{p}_{n+1}^{(1)}$, for $n = 2, 3, 4$; $a = 1.5, 2, 2.5$. Table 2 summarizes these results and shows that the values of these predictions are close to the corresponding true values of the mean. This example shows that the recurrence relations, defined in Theorems 2.3, 2.4 and 4.1, work well with the estimates of the moments whenever these estimates are efficient (unbiased and consistent).

6. Discussion

The moments of current records and record range have considerable importance in many applications. In this paper we derived some recurrence relations satisfied by the single as well as product moments of current records and record range, based on the Weibull, Pareto and exponential cdf's, primarily to reduce the amount of direct computations. In addition, these recurrence relations usefully express the higher order moments in terms of the lower order moments and hence make the evaluation of higher order moments easy. Moreover, they are very useful in checking the computation of these moments. Most of the known recurrence relations in the literature satisfy the preceding desirable properties, but to the best of our knowledge a few well-known recurrence relations are predictive recurrence relations. A recurrence relation $\theta_{n+1} = F(\theta_{i_1}, \theta_{i_1}, \ldots, \theta_{i_k})$, $1 \le i_1 < i_2 < \cdots < i_k \le n$, satisfied by the sequence θ_n is said to be predictive recurrence relation, if for any unbiased and consistent estimates $\bar{\theta}_{i_1}, \bar{\theta}_{i_2}, \ldots, \bar{\theta}_{i_k}$ of the parameters $\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_k}$, the prediction estimate $\hat{\theta}_{n+1} = F(\bar{\theta}_{i_1}, \bar{\theta}_{i_2}, \ldots, \bar{\theta}_{i_k})$ is unbiased and consistent estimate for θ_{n+1}. Clearly any recurrence relation which has the simple form $\theta_{n+1} = a\theta_n + b$, where $a \neq 0$ and b are constants w.r.t. the sequence θ_n, is a predictive recurrence relation. It is worth noting that, the recurrence relations defined in Theorems 2.3, 2.4 and 4.1 having this simple form and consequently they serve to be predictive recurrence relations.

Table 2
True, estimated and predicted mean of the upper current record for Pareto cdf.

$a = 1.5$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
True mean $p_n^{(1)}$	4.5	6.75	10.125	15.1875
Estimated mean $\bar{p}_n^{(1)}$	4.5038	6.4340	10.3592	15.2559
Predicted mean $\hat{p}_{n+1}^{(1)}$	–	6.7557	9.6510	15.5388
$a = 2$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
True mean $p_n^{(1)}$	2.67	3.56	4.74	6.32
Estimated mean $\bar{p}_n^{(1)}$	2.7537	3.6052	4.7012	6.0888
Predicted mean $\hat{p}_{n+1}^{(1)}$	–	3.6716	4.8069	6.2683
$a = 2.5$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
True mean $p_n^{(1)}$	2.0833	2.6042	3.2552	4.0690
Estimated mean $\bar{p}_n^{(1)}$	2.2184	2.4532	3.1204	4.2734
Predicted mean $\hat{p}_{n+1}^{(1)}$	–	2.7730	3.0665	3.9005

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References

Ahmadi, J., & Balakrishnan, N. (2004). Confidence intervals for quantiles in terms of record range. *Statistics & Probability Letters*, *68*, 1955–1963.

Ahmadi, J., & Balakrishnan, N. (2005). Preservation of some reliability properties by certain record statistics. *Statistics*, *39*, 347–354.

Ahmadi, J., & Balakrishnan, N. (2008). Prediction intervals for future records. *Statistics & Probability Letters*, *78*, 395–405.

Ahmadi, J., & Balakrishnan, N. (2011). Distribution-free prediction intervals for order statistics based on record coverage. *Journal of the Korean Statistical Society*, *40*, 181–192.

Ahmadi, J., Razmkhaha, M., & Balakrishnan, N. (2009). Current k-records and their use in distribution-free confidence intervals. *Statistics & Probability Letters*, *79*, 29–37.

Arnold, B. C., Balakrishnan, N., & Nagaraja, H. N. (1998). *Records*. New York: Wiley.

Barakat, H. M. (2010). Continuability of local weak limits for record values. *Statistics*, *44*(3), 269–274.

Basak, P. (2000). An application of record range and some characterization results. In N. Balakrishnan (Ed.), *Advances on theoretical and methodology aspects of probability and statistics* (pp. 83–95). New york: Gordon and Breach Science Publishers.

Galambos, J. (1987). *The asymptotic theory of extreme order statistics* (2nd ed.). FL: Krieger.

Houchens, R.L. (1984). Record value, theory and inference, Ph.D. dissertation, University of California, Riverside, CA.

Jankov, D. (2011). Integral expressions for series of functions of hypergeometric and Bessel types, Ph.D. dissertation, University of Zagreb.

Johnson, N. L., & Kotz, S. (1970). *Continuous univariate ditributions-1*. Boston: Houghton Mifflin Company.

Raqab, M. R. (2009). Distribution-free prediction intervals for the future current record statistics. *Statistical Papers*, *50*, 429–439.

Resnik, S. I. (1973). Limit laws for record values. *Stochastic Processes and their Applications*, *1*, 67–82.

Smirnov, N. V. (1952). Limit distributions for the terms of a variational series. *American Mathematical Society Translations, Series 1*, *No. 67*, Original published in 1949.

Wu, C. Y. (1966). The types of limit distributions for terms of variational series. *Scientia Sinica*, *15*, 749–762.