Motion in Bimetric Type Theories of Gravity

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Abstract

The problem of motion for different test particles , charged and spinning objects of constant spinning tensor in different versions of bimetric theory of gravity is obtained by deriving their corresponding path and path deviation equations, using a modified Bazanski in presence of Riemannian geometry. This method enables us to find path and path deviation equations of different objects orbiting very strong gravitational fields.

1 Bimetric Theories: A brief introduction

General relativity is considered a land mark in history of science of being, during the last century, a pioneer non-linear theory of gravity[1]. Yet, some problems have been remained unsolved due to applying Riemannian geometry in its explanation. One of these difficulties is related to the law of conservation of energy and momentum [2], which is dealt by considering the metrical tensor as flat one at great distances from the gravitational source. Rosen [3,4]introduced a remedy to this problem by proposing two different metrics $g_{\mu\nu}$ representing the gravitational source and giving a curved space and $\gamma_{\mu\nu}$ describing a physical one expressing an inertial frame and becoming a flat space. Using this assumption, it can be found that the field equations of Einstein imply to a theory of gravitation in flat space . A problem arises because the pseudo-tensor quantities in Orthodox General Relativity then turn out to be tensorial ones [2]. This led Yalmoz [5] to examine a new class of solutions for the field equations of Rosen's theory of gravitation to solve the difficulty of dealing with the flat metric. But, a slight problem has emerged in that, the speed of light is no longer constant as confirmed experimentally in the realm of Special Relativity. Such a problem is counted to be a virtue by expressing Moffat's approach of a bi-metric theory of gravity for a variable speed of light, as it helps to reveal the puzzle of dark energy problem. This approach may be expressed by means of two metrics are likened to each other in terms of gradients of scalar /biscalar fields to explain the rapid expansion of galaxies is due to change of speed of light from one epoch to another that

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i.e. dark energy can be interpreted This type of theories are expressing how inflation senecio of the universe is due to Bi-metric version of variable speed of light [7] . Some applications using the Moffat formalism of bi-metric theory of gravity are explaining the causal description of quantum entanglement [8] and testing the propagation of neutrinos using OPERA experiment [9].

Another problem in GR is inability to explain the rotation curves of spiral galaxies. Milgram [10] proposed a specific treatment by performing a modified Newtonian Dynamics paradigm (MOND)that was able to explain its causality, apart from appealing to dark matter problem. A new step has been taken successively to extend MOND to be expressed in terms of bi-metric theory to be produce BIMOND [11] by having two field equations describing matter and twin-matter[12] may be used for examining the existence of gravitational waves and explaining two interacting 4D membranes[13]. Recently, Hassan-Rosen [14] developed an extension of the present bi-metric theory using the concept of bi-gravity, which used two metrics describing their gravitational fields. In doing so, they discard the earlier bimetric theories metrics in which one metric describes gravity while the other is physical. Accordingly, a new massive gravity theory for spin-2 , free from ghosts, has been obtained [15]. Several applications of these types of theories are viable in dealing with obtaining field equations for very strong gravitational fields of neutron stars [16], which suggests the possibility to examine black holes and super-massive black holes e.g Sgr A* by studying the stability of objects orbiting in these fields. This hypothesis is essentially to be examined.

2 Motion in Bimetric theories

Theories of gravity may help to explain the functioning of different objects. For example, Rosen[3] obtained the equation of motion in bimetric theory of gravitation for a test particle. These equations were solved by Isrealit[17] in order to examine their behavior in the case of small velocities and weak fields using past Newtonian approximation .

Also, Falik and Opher [18] applied the bimetric theory of gravity to find the field equations associated with spinning neutron stars as an example of strong gravitational field . This achievement opens the way to examine motion of different charged objects and spinning ones in presence of a strong gravitational field as defined by Bi-metric type theories. This led us to obtain the corresponding path and path deviation of different objects , such as test particles, charged particle, spinning objects. These results are determined by introducing a Lagrangian with a specific feature for obtaining the path and path deviations. Therefore, the key role to this approach is based to obtain path and path deviation equations for each of these objects using a specified Lagrangian for each case. The aim of our study is obtaining equations of motion for objects using bimetric theory originated from two metrics in one stands for gravitational field and the other defines physical matter or both represent gravity.

2.1 Path and Path Deviation Equations: The Bazanski Approach

Geodesic and geodesic deviation equations can be obtained from the following Bazanski Lagrangian [19]:

$$L = g_{\alpha\beta} U^{\alpha} \frac{D\Psi^{\beta}}{Ds}, \tag{1}$$

where U^{α} is a four vector velocity, Ψ^{β} its deviation vector, $\frac{D}{DS}$ is a covariant derivative with respect to $g_{\mu\nu}$. If one takes the variation with respect to the deviation vector Ψ^{ρ} to obtain geodesic equations:

$$\frac{dU^{\alpha}}{dS} + \Gamma^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = 0, \tag{2}$$

where $\Gamma^{\alpha}_{\mu\nu}$ is the Levi-Civita affine connection. Also, the same technique can be applied to obtain the variation with respect to the tangent vector U^{ρ} to obtain the geodesic deviation equations:

$$\frac{D\Psi^{\alpha}}{DS^{2}} = R^{\alpha}_{.\beta\gamma\delta} \Psi^{\gamma} U^{\beta} U^{\delta} \tag{3}$$

Where $R^{\alpha}_{.\beta\gamma\delta}$ the curvature of space time defined by the affine connection $\Gamma^{\alpha}_{\mu\nu}$. The above method has been applied in different geometries than the Riemannian one e.g. non-Riemannian geometries admitting non-vanishing curvature and torsion tensors simultaneously [20-22]. This approach helps to implement the concept of geometrization to include not only physics but also biological epidemic curves [23] as well as economic complex systems in terms of information geometry [24]. Also, this Lagrangian has been modified to describe the path equation of charged object to take the following form [25];

$$L = g_{\alpha\beta} U^{\alpha} \frac{D\Psi^{\beta}}{Ds} + \frac{e}{m} F_{\alpha\beta} U^{\alpha} \Psi^{\beta} \tag{4}$$

to give

$$\frac{dU^{\alpha}}{d\bar{S}} + \Gamma^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = \frac{e}{m} F^{\mu}_{.\nu} U^{\nu} \tag{5}$$

where $F_{\mu\nu}$ is an electromagnetic tensor, $\frac{e}{m}$ the ratio between charge to mass of an object. and its corresponding deviation equation becomes:

$$\frac{D^{2}\Psi^{\alpha}}{DS^{2}} = R^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{e}{m}(F^{\alpha.\nu} \frac{D\Psi^{\nu}}{Ds} + F^{\alpha}_{.\nu;\rho} U^{\nu} \Psi^{\rho}). \tag{6}$$

In the mean time the corresponding Papapetrou Equation for spinning objects with constant spinning tensor [26] is obtained from the following Lagrangian :

$$L = g_{\alpha\beta} U^{\alpha} \frac{D\Psi^{\beta}}{DS} + \frac{1}{2m} R_{\alpha\beta\gamma\sigma} U^{\alpha} \Psi^{\beta} S^{\gamma\sigma} \tag{7}$$

where $S^{\mu\nu}$ is a spin tensor of a spinning object. By taking variation with respect to the Ψ^{α} to obtain

$$\frac{dU^{\alpha}}{dS} + \Gamma^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = \frac{1}{2} R^{\alpha}_{.\mu\nu\rho} S^{\rho\nu} U^{\mu} \tag{8}$$

and taking the variation with respect to U^α to obtain its deviation equation:

$$\frac{D^2\Psi^\alpha}{DS^2} = R^\alpha_{.\mu\nu\rho} U^\mu U^\nu \Psi^\rho + \frac{1}{2m}(R^\alpha_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^\nu}{Ds} + R^\alpha_{\mu\nu\lambda} S^{\mu\lambda}_{.;\rho} U^\nu \Psi^\rho + R^\alpha_{\mu\nu\lambda;\rho} S^{\nu\lambda} U^\mu \Psi^\rho), \qquad (9)$$

as well as the Dixon equation for spinning charged objects is expressed as [27]

$$\frac{dU^\alpha}{dS} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = \frac{e}{m} F^\mu_{.\nu} U^\nu + \frac{1}{2m} R^\alpha_{.\mu\nu\rho} S^{\rho\nu} U^\mu, \qquad (10)$$

while, its corresponding deviation equation becomes

$$\frac{D^2\Psi^\alpha}{Ds^2} = R^\alpha_{.\mu\nu\rho} U^\mu U^\nu \Psi^\rho + \frac{e}{m}(F^{\alpha.\nu} \frac{D\Psi^\nu}{Ds} + F^\alpha_{.\nu;\rho} U^\nu \Psi^\rho) + \frac{1}{2m} R^\alpha_{.\mu\nu\rho} U^\mu U^\nu \Psi^\rho$$
$$+ \frac{1}{2m}(R^\alpha_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^\nu}{Ds} + R^\alpha_{\mu\nu\lambda} S^{\mu\lambda}_{.;\rho} U^\nu \Psi^\rho + R^\alpha_{\mu\nu\lambda;\rho} S^{\nu\lambda} U^\mu \Psi^\rho) \qquad (11)$$

The Papapetrou equation of a spinning object with precession [28]is obtained by a modified Bazanski Lagrangian [29] :

$$L = g_{\alpha\beta}(mU^\alpha + U_\beta \frac{DS^{\alpha\beta}}{DS}) \frac{D\Psi^\beta}{Ds} + \frac{1}{2} R_{\alpha\beta\gamma\delta} S^{\gamma\delta} U^\beta \Psi^\alpha \qquad (12)$$

to obtain equation of a spinning object by taking the variation with respect to the deviation vector Ψ^α

$$\frac{D}{DS}(mU^\alpha + U_\beta \frac{DS^{\alpha\beta}}{DS}) = \frac{1}{2} R^\alpha_{.\mu\nu\rho} S^{\rho\nu} U^\mu \qquad (13)$$

and its deviation equation can be obtained by taking the variation with respect to U^α to become:

$$\frac{D^2\Psi^\alpha}{Ds^2} = R^\alpha_{.\mu\nu\rho} U^\mu (mU^\nu + U_\beta \frac{DS^{\nu\beta}}{Ds}) \Psi^\rho + g^{\alpha\sigma} g_{\nu\lambda} (mU^\lambda + U_\beta \frac{DS^{\lambda\beta}}{Ds})_{;\sigma} \frac{D\Psi^\nu}{Ds}$$
$$+ \frac{1}{2}(R^\alpha_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^\mu}{Ds} + R^\alpha_{\mu\nu\lambda} S^{\nu\lambda}_{.;\rho} U^\mu \Psi^\rho + R^\alpha_{\mu\nu\lambda;\rho} S^{\nu\lambda} U^\mu \Psi^\rho). \qquad (14)$$

2.2 Path and Path Deviation Equations in Weyl geometry

It is well known that in Weyl geometry the gravitational potential tensor is associated with such a scalar field. From this perspective one can define a combined gravitation potential tensor in the following manner [30]:

$$\bar{g}^{\mu\nu} = e^\phi g^{\mu\nu}, \qquad (15)$$

Where $\bar{g}_{\mu\nu}$ is the Weyl gravitational potential, and ϕ a scalar field, which may give raise to introduce disformal transformation of any gravitational theory having two metrics defined in the following way [31]

$$\bar{g}_{\mu\nu} = [A g_{\mu\nu} + \bar{A} \phi_{,\mu} \phi_{,\nu}] \qquad (16)$$

where A and $\bar{A}$ are arbitrary constants .

Thus, in this type of geometry it can be defined its corresponding affine connection to become :

$$\bar{\Gamma}^{\alpha}_{\beta\sigma} = \Gamma^{\alpha}_{\beta\sigma} + \frac{1}{2} g^{\alpha\delta}(g_{\sigma\delta}\phi_{,\beta} + g_{\delta\beta}\phi_{,\sigma} - g_{\beta\sigma}\phi_{,\delta}) \tag{17}$$

In order to obtain the geodesic equation in the following which can be obtained by applying the action principle on the following Lagrangian:

$$L = \bar{g}_{\mu\nu} U^{\mu} U^{\nu} \tag{18}$$

to give

$$\frac{dU^{\alpha}}{dS} + \bar{\Gamma}^{\alpha}_{\beta\sigma} U^{\beta} U^{\sigma} = 0, \tag{19}$$

which is obtained by taking the variation with respect to Ψ^{μ} on its a developed Lagrangian mentioned in [32] :

$$L = \bar{g}_{\mu\nu} U^{\mu} (\frac{d\Psi^{\nu}}{dS} + \bar{\Gamma}^{\nu}_{\rho\delta} \Psi^{\rho} U^{\delta}). \tag{20}$$

And its corresponding deviation equation is obtained by taking the variation with respect to U^{μ} to become:

$$\frac{\bar{D}^2 \Psi^{\alpha}}{\bar{D}S^2} = \bar{R}^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} \tag{21}$$

where

$$\bar{R}^{\alpha}_{.\mu\nu\rho} = \bar{\Gamma}^{\alpha}_{\mu\rho,\nu} - \bar{\Gamma}^{\alpha}_{\mu\nu,\rho} + \bar{\Gamma}^{\sigma}_{\mu\rho} \bar{\Gamma}^{\alpha}_{\sigma\rho} - \bar{\Gamma}^{\sigma}_{\mu\rho} \bar{\Gamma}^{\alpha}_{\sigma\rho}$$

(ii) Dixon-like Equation for spinning charged objects of Weyl geometry: Similarely, we can obtain the Dixon -like path equation

$$\frac{dU^{\alpha}}{dS} + \bar{\Gamma}^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = \frac{e}{m} F^{\mu}_{.\nu} U^{\nu} + \frac{1}{2m} \bar{R}^{\alpha}_{.\mu\nu\rho} S^{\rho\nu} U^{\mu} \tag{22}$$

and its corresponding deviation equation becomes:

$$\frac{\bar{D}^2 \Psi^{\alpha}}{\bar{D}S^2} = \bar{R}^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{e}{m}(F^{\alpha.\nu} \frac{\bar{D}\Psi^{\nu}}{\bar{D}S} + F^{\alpha}_{.\nu;\rho} U^{\nu} \Psi^{\rho}) + \frac{1}{2m} \bar{R}^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho}$$
$$+ \frac{1}{2m}(\bar{R}^{\alpha}_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^{\nu}}{Ds} + \bar{R}^{\alpha}_{\mu\nu\lambda} S^{\mu\lambda}_{.;\rho} U^{\nu} \Psi^{\rho} + \bar{R}^{\alpha}_{\mu\nu\lambda;\rho} S^{\nu\lambda} U^{\mu} \Psi^{\rho}) \tag{23}$$

2.3 Path and Path Deviation Equations of MOND

In this part, it is worth mentioning the path and path deviation of MOND paradigm due to its vital role in explaining the vague regions due to dark matter problem that are unknown by Newtonian/Einsteinain formulations i.e. revealing the nature of rotation curves of spiral galaxies [33] Accordingly, some authors have studied motion of a test in MOND [34]. This has led us to apply the Bazanski method in order to obtain the path and path deviation equations for any test particle related to this paradigm.

$$L = g_{\mu\nu} U^{\alpha} \frac{D\Psi^{\beta}}{Ds} + \frac{1}{2m} \phi_{,\mu} \Psi^{\mu} \tag{24}$$

to give its path equation

$$\frac{dU^{\alpha}}{dS} + \Gamma^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = \frac{1}{2m} g^{\alpha\mu} \phi_{,\mu} \qquad (25)$$

and its deviation equation in the following way:

$$\frac{D^2\Psi^{\alpha}}{DS^2} = R^{\alpha}_{\beta\gamma\delta} U^{\beta} U^{\gamma} \Psi^{\delta} + g^{\alpha\rho}\phi_{\rho;\sigma}\Psi^{\sigma} + g^{\alpha\rho}\phi_{\rho} U_{\nu} \frac{D\Psi^{\nu}}{DS} \qquad (26)$$

3 Path and Path deviation of Bimetric Theories

3.1 Path Equation and Path Deviation of Rosen's Approach

Equations of motion of test particles subject to bimetric theory of gravity were obtained by Rosen [4] in the following way:

$$\frac{dU^{\mu}}{dS} + \Delta^{\mu}_{\nu\sigma} U^{\nu} U^{\sigma} = 0, \qquad (27)$$

where

$$\Delta^{\mu}_{\nu\sigma} = [\Gamma^{\mu}_{\nu\sigma} - \gamma^{\mu}_{\nu\sigma}],$$

and $\gamma^{\mu}_{\nu\sigma}$ is an affine connection defined by $\gamma_{\mu\nu}$.
Following the Rosen approach, Isrealit (1976) solved this type of motion using PPN approximation to be compared with the previous findings in GR. Lately, Foukzon et al [35] studied the bimetric theory of gravitational inertial field in Riemannain and its relationship with Finsler-Lagrange geometry and obtained the same path equation, which can be obtained from the following Lagrangian:

$$L = (g_{\mu\nu} - \gamma_{\mu\nu}) U^{\mu} U^{\nu}. \qquad (28)$$

In an alternative way, we suggest its corresponding lagrangian which follows the Bazanki Lagrangian to obtain the sets of equations of geodesic and geodesic deviation by taking the ovation with respect to Ψ to obtain

$$L = (g_{\mu\nu} - \gamma_{\mu\nu}) U^{\mu} \frac{D\Psi^{\nu}}{DS} \qquad (29)$$

While, taking the variation with respect U^{α} we obtain

$$\frac{\nabla^2\Psi^{\nu}}{\nabla S^2} = (R^{\alpha}_{\beta\gamma\sigma} - P^{\alpha}_{\beta\gamma\sigma}) U^{\beta} U^{\gamma} \Psi^{\sigma}$$

where

$$\frac{\nabla\Psi^{\nu}}{\nabla S} = \frac{d\Psi^{\mu}}{dS} + \Delta^{\mu}_{\nu\sigma} \Psi^{\nu} U^{\sigma}$$

and $P^{\alpha}_{\beta\gamma\sigma}$ is the curvature tensor obtained by the affine connection $\gamma^{\alpha}_{\beta\delta}$ [3].

Due to Rosen's approach the curvature tensor $P^{\alpha}_{\beta\gamma\sigma} = 0$ which let equation (..) reduces to

$$\frac{\nabla^2\Psi^{\nu}}{\nabla S^2} = R^{\alpha}_{\beta\gamma\sigma} U^{\beta} U^{\gamma} \Psi^{\sigma}.$$

Also, for charged objects in bimetric theory of gravity, Falik and Rosen [36] obtained their corresponding field equations , which led us to introduces the following Lagrangian to obtain their corresponding path and path deviation equations:

$$L = (g_{\mu\nu} - \gamma_{\mu\nu}) U^{\mu} \frac{\nabla \Psi^{\nu}}{\nabla S} + \frac{e}{m} F_{\mu\nu} U^{\mu} \Psi^{\nu} \tag{30}$$

to give

$$\frac{\nabla U^{\alpha}}{\nabla S} = \frac{e}{m} F^{\alpha}_{.\nu} U^{\nu} \tag{31}$$

and its corresponding deviation equation becomes:

$$\frac{\nabla^2 \Psi^{\alpha}}{\nabla S^2} = R^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{e}{m} F^{\alpha}_{.\nu} \frac{\nabla \Psi^{\nu}}{\nabla S} + \frac{e}{m} (F^{\alpha}_{.\nu;\rho} - F^{\alpha}_{.\nu|\rho}) U^{\nu} \Psi^{\rho} \tag{32}$$

Moreover, Avakian et al.[16] studied the field equations of a spinning bodies in the presence of Bimetric theory. Accordingly its the corresponding spinning equation can be obtained from the following Lagrangian:

$$L = (g_{\alpha\beta} - \gamma_{\alpha\beta}) U^{\alpha} \frac{\nabla \Psi^{\beta}}{\nabla S} + \frac{1}{2m} (R_{\alpha\beta\gamma\sigma} - P_{\alpha\beta\gamma\sigma}) U^{\alpha} \Psi^{\beta} S^{\gamma\sigma} \tag{33}$$

we can apply its Bazanski approach to obtain its path equation:

$$\frac{dU^{\alpha}}{dS} + \Delta^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = \frac{1}{2m} (R^{\alpha}_{.\mu\nu\rho} - P^{\alpha}_{.\mu\nu\rho}) S^{\rho\nu} U^{\mu} U^{\nu} \tag{34}$$

and its corresponding deviation equation:

$$\begin{aligned} \frac{\nabla^2 \Psi^{\alpha}}{\nabla S^2} &= (R^{\alpha}_{.\mu\nu\rho} - P^{\alpha}_{.\mu\nu\rho}) U^{\mu} U^{\nu} \Psi^{\rho} + \frac{1}{2m} (R^{\alpha}_{.\mu\nu\rho} - P^{\alpha}_{.\mu\nu\rho}) S^{\nu\rho} \frac{D\Psi^{\nu}}{Ds} \\ &\quad + (R^{\alpha}_{\mu\nu\lambda} S^{\mu\lambda}_{.;\rho} - P^{\alpha}_{\mu\nu\lambda} S^{\mu\lambda}_{.|\rho}) U^{\nu} \Psi^{\rho} + (R^{\alpha}_{\mu\nu\lambda;\rho} - P^{\alpha}_{\mu\nu\lambda|\rho}) S^{\nu\lambda} U^{\mu} \Psi^{\rho} \end{aligned} \tag{35}$$

Thus, if we take into consideration that $P^{\alpha}_{\beta\gamma\delta} = 0$, path equation becomes:

$$\frac{dU^{\alpha}}{dS} + \Delta^{\alpha}_{\mu\nu} U^{\mu} U^{\nu} = \frac{1}{2m} R^{\alpha}_{.\mu\nu\rho} S^{\rho\nu} U^{\mu} U^{\nu} \tag{36}$$

and its corresponding deviation equation becomes:

$$\frac{\nabla^2 \Psi^{\alpha}}{\nabla S^2} = R^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{1}{2m} R^{\alpha}_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^{\nu}}{Ds} + \frac{1}{2m} R^{\alpha}_{\mu\nu\lambda} (S^{\mu\lambda}_{.;\rho} - S^{\mu\lambda}_{.|\rho}) U^{\nu} \Psi^{\rho} + \frac{1}{2m} R^{\alpha}_{\mu\nu\lambda;\rho} S^{\nu\lambda} U^{\mu} \Psi^{\rho} \tag{37}$$

3.2 Path and Path Deviation Equations of Moffat's Approach

Moffat [6] presented the framework of VSL satisfying bimetric theory and its causality to reveal the problem of dark energy due to VSL by introducing such a metric in the following way.

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + B\partial_\mu\phi\partial_\nu\phi \tag{38}$$

where $\hat{g}_{\mu\nu}$ defines a specific matter metric tensor of a given matter field, B is an arbitrary constant has a dimension of $[length]^2$ and chosen to be positive and ϕ is a biscalar field . The inverse metrics $g^{\mu\nu}$ $\hat{g}^{\mu\nu}$ satisfy

$$g^{\mu\nu}g_{\mu\rho} = \delta^\nu_\rho \tag{39}$$

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + B\partial_\mu\phi\partial_\nu\phi \tag{40}$$

$$\hat{g}^{\mu\nu}\hat{g}_{\mu\rho} = \delta^\nu_\rho \tag{41}$$

Yet, the modification processes to control the casual propagation of the biscalar field led to redefine (40) to become:

$$\hat{g}^{\mu\nu} = g^{\mu\nu} + \frac{B}{K}\nabla_\mu\phi\nabla_\nu\phi + KB\sqrt{T_{\mu\nu}}, \tag{42}$$

where K is an arbitrary constant and $T_{\mu\nu}$ is a given energy-momentum tensor to control the causal propagation of the biscalar field [9].

Consequently, the equation of geodesic can be obtained from taking the action on the following Lagrangian

$$L = \hat{g}_{\mu\nu}U^\mu U^\nu \tag{43}$$

to become

$$\frac{dU^\nu}{dS} + \hat{\Gamma}^\nu_{\mu\rho}U^\rho U^\mu = 0, \tag{44}$$

where

$$\hat{\Gamma}^\nu_{\mu\rho} = \frac{1}{2}\hat{g}^{\sigma\nu}(\hat{g}_{\rho\sigma,\mu} + \hat{g}_{\mu\sigma,\rho} - \hat{g}_{\rho\mu,\sigma})$$

Also, we suggest its corresponding lagrangian which follows the Bazanki lagrangian to obtain the sets of equations of geodesic and geodesic deviation by taking the ovation with respect to Ψ^α to obtain (44)

$$L = \hat{g}_{\mu\nu}U^\mu\frac{\hat{D}\Psi^\nu}{\hat{D}S} \tag{45}$$

While, taking the variation with respect to U^α to obtain:

$$\frac{\hat{D}^2\Psi^\nu}{\hat{D}S^2} = \hat{R}^\alpha_{\beta\gamma\delta}U^\gamma U^\beta\Psi^\delta$$

where

$$\hat{R}^\alpha_{\beta\gamma\delta} = \hat{\Gamma}^\alpha_{\beta\delta,\gamma} - \hat{\Gamma}^\alpha_{\beta\gamma,\delta} + \hat{\Gamma}^\nu_{\beta\delta}\hat{\Gamma}^\alpha_{\nu\gamma} - \hat{\Gamma}^\nu_{\beta\gamma}\hat{\Gamma}^\alpha_{\nu\delta}$$

3.3 Path equations and Path deviation of BIMOND Type Theories

In this section, we present the corresponding path and path deviation equation for test particles or spinning objects in the presence of BIMOND theories. Accordingly , it is worth mentioning at the beginning the above corresponding paths and there deviation equation in MOND paradigm to be extended in case of its BIMOND version its corresponding path equation becomes:

$$\frac{dU^{\alpha}}{dS} + (\Gamma^{\alpha}_{\beta\gamma} - \bar{\Gamma}^{\alpha}_{\beta\gamma}) U^{\beta} U^{\gamma} = 0$$

which can be formed from the following lagrangian by taking the variation with respect to Ψ^{α}:

$$L = \hat{g}_{\mu\nu} \frac{\hat{D}\Psi^{\nu}}{Ds}$$

where:

$$\frac{D\Psi^{\alpha}}{DS} = \frac{d\Psi^{\alpha}}{ds} + (\Gamma^{\alpha}_{\beta\gamma} - \bar{\Gamma}^{\alpha}_{\beta\gamma}) \Psi^{\beta} U^{\gamma}$$

and its corresponding deviation equation becomes;

$$\frac{D^2\Psi^{\alpha}}{DS^2} = (R^{\alpha}_{\beta\gamma\delta} - \bar{R}^{\alpha}_{\beta\gamma\delta}) \Psi^{\gamma} U^{\delta} U^{\beta}$$

In case of BIMOND , Milgram [14] introduced the relationship between the two affine connections as defined by $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ to become:

$$C^{\alpha}_{\beta\rho} = \Gamma\alpha_{\beta\rho} - \bar{\Gamma}\alpha_{\beta\rho},$$

such that [,,,]:

$$, g_{\mu\nu;\rho} = g_{\delta\nu} C^{\delta}_{\mu\rho} + g_{\delta\mu} C^{\delta}_{\nu\rho}$$

and

$$\gamma_{\mu\nu|\rho} = -\gamma_{\delta\nu} C^{\delta}_{\mu\rho} - \gamma_{\delta\mu} C^{\delta}_{\nu\rho}.$$

Thus, we suggest the following Lagrangian path and path deviation equations becomes

$$L = g_{\mu\nu} U^{\mu} \frac{\bar{D}^2\Psi^{\nu}}{DS^2} + \gamma_{\mu\nu} U^{\mu} \frac{D^2\Phi^{\nu}}{D\tau^2} + \frac{1}{2m} (R_{\alpha.\mu\nu\rho} - \bar{R}_{\alpha.\mu\nu\rho}) S^{\nu\rho} U^{\mu} \Psi^{\alpha}$$

In case of spinning object

$$\frac{DU^{\alpha}}{DS} = \frac{1}{2m} (R^{\alpha}_{.\mu\nu\rho} - \bar{R}^{\alpha}_{.\mu\nu\rho}) S^{\rho\nu} U^{\mu} \tag{46}$$

and its deviation equation can be obtained by taking the variation with respect to U^{α} to become:

$$\frac{D^2\Psi^{\alpha}}{DS^2} = R^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{1}{2m} (R^{\alpha}_{.\mu\nu\rho} - \bar{R}^{\alpha}_{.\mu\nu\rho})$$

$$+ \frac{1}{2m} R^{\alpha}_{.\mu\nu\rho} S^{\nu\rho} \frac{D\Psi^{\nu}}{Ds} + (R^{\alpha}_{.\mu\nu\rho} S^{\mu\lambda}_{.;\rho} U^{\nu} \Psi^{\rho} - \bar{R}^{\alpha}_{.\mu\nu\rho} S^{\mu\lambda}_{.|\rho} U^{\nu} \Psi^{\rho}) + (R^{\alpha}_{\mu\nu\lambda;\rho} - \bar{R}^{\alpha}_{\mu\nu\lambda|\rho}) S^{\nu\lambda} U^{\mu} \Psi^{\rho} \tag{47}$$

3.4 Generalized Path equations and Path Deviation of Bimetric Theories

Hossenfelder [35] introduced an alternative version of bi-metric theory, having two different metrics **g** and **h** of Lorentzian signature on a manifold **M** one is defined in tangential space TM and the other is in its co-tangential space T*M respectively. These can be regarded as two sorts of matter and twin matter, existing individually , each of them has its own field equations as defined within Riemannian geometry. In this part we are gong to present a generalized form which can be present different types of path and path deviation which can be explained for any bimetric theory which has two different metrics and curvatures as defined by Riemannian geometry . Their Corresponding Lagrangian can be expressed in the following way:

$$L = g_{\mu\nu} \Psi_{;\nu} U^{\nu} + \gamma_{\mu\nu} \Phi_{|\nu} V^{\mu}, \tag{48}$$

By considering $\frac{d\tau}{ds} = 0$ it will lead to two separate sets of path equations owing to each parameter by applying the following Bazanski-like lagrangian:

$$L = g_{\mu\nu} \Psi_{;\nu} U^{\nu} - \gamma_{\mu\nu} \Phi_{|\nu} V^{\mu} V^{\nu}$$

$$\frac{DU^{\alpha}}{DS} = 0, \tag{49}$$

and

$$\frac{DV^{\alpha}}{D\tau} = 0 \tag{50}$$

and their corresponding path deviation equations:

$$\frac{D^2 \Psi^{\alpha}}{DS^2} = R^{\alpha}_{\beta\gamma\delta} U^{\gamma} U^{\beta} \Psi^{\delta}, \tag{51}$$

and

$$\frac{D^2 \Phi^{\alpha}}{D\tau^2} = S^{\alpha}_{\beta\gamma\delta} V^{\gamma} V^{\beta} \Phi^{\delta}, \tag{52}$$

Thus we suggest, the corresponding lagrangian to describe two independent sets of a generalized path and path deviation equations:

$$L = g_{\mu\nu} \Psi_{;\nu} U^{\nu} - \gamma_{\mu\nu} \Phi_{|\nu} V^{\mu} V^{\nu} + f_{\mu} \Psi^{\mu} + \hat{f}_{\mu} \Phi^{\mu} \tag{53}$$

where,

$$f_{\mu} = \frac{1}{m}(eF_{\mu\nu} + \frac{1}{2} R_{\mu\nu\rho\sigma} S^{\rho\sigma}) U^{\nu}$$

and

$$\bar{f}_{\mu} = \frac{1}{m}(eF_{\mu\nu} + \frac{1}{2} S_{\mu\nu\rho\sigma} S^{\rho\sigma}) V^{\nu}$$

. By taking the variation of Ψ^{α} and Φ^{α} we obtain the generalized set of path deviation

$$\frac{DU^{\alpha}}{DS} = f^{\alpha}, \tag{54}$$

and

$$\frac{DV^{\alpha}}{D\tau} = \bar{f}^{\alpha} \tag{55}$$

and taking the variation with respect to U^{α} and V^{α} to obtain the set of their corresponding path deviation equations:

$$\frac{D^2\Psi^{\alpha}}{DS^2} = R^{\alpha}_{\beta\gamma\delta} U^{\gamma} U^{\beta} \Psi^{\delta} + f^{\alpha}_{;\rho} \Psi^{\rho} + g^{\alpha\rho} f_{\rho} U_{\nu} \frac{D\Psi^{\nu}}{DS} \tag{56}$$

and

$$\frac{D^2\Phi^{\alpha}}{D\tau^2} = S^{\alpha}_{\beta\gamma\delta} V^{\gamma} V^{\beta} \Phi^{\delta} + \bar{f}^{\alpha}_{|\rho} \Phi^{\rho} + \gamma^{\alpha\rho} \bar{f}_{\rho} V_{\nu} \frac{D\Psi^{\nu}}{D\tau} \tag{57}$$

3.5 Path Equations and Path Deviation of Bi-gravity Type Theories

Recently, Arkami et al[30] have suggested two independent metrics to explain bi-gravity phenomena,

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

and

$$d\tau^2 = h_{\mu\nu} dx^{\mu} dx^{\nu}$$

.

Thus, the variational method to obtain geodesic-like equations of bigravity theory is expressed in the following way [36] :

$$\left(\frac{d}{dS} \frac{\partial L}{\partial \dot{\Phi}^{\alpha}} - \frac{\partial L}{\partial \Phi^{\alpha}}\right) + \left(\frac{d\tau}{dS}\right)^2 \left(\frac{d}{d\tau} \frac{\partial L}{\partial \bar{\bar{\Phi}}^{\alpha}} - \frac{\partial L}{\partial \Phi^{\alpha}}\right) = 0$$

to give the same results as mentioned by Arkani et al (2014)

$$g_{\mu\nu} \frac{DU^{\mu}}{DS} + h_{\mu\nu} \left(\frac{d\tau}{dS}\right) \frac{DU^{\mu}}{D\tau} = 0 \tag{58}$$

Applying the same technique of the Bazanski approach, we obtain its deviation equations to obtain:

$$g_{\mu\alpha}\left[\frac{D^2\Psi^{\alpha}}{DS^2} + R^{\alpha}_{\beta\delta\gamma} U^{\gamma} U^{\beta} \Psi^{\delta}\right] + \left(\frac{d\tau}{dS}\right)^2 \gamma_{\mu\alpha}\left[\frac{D^2\Phi^{\alpha}}{D\tau^2} + R^{\alpha}_{\beta\delta\gamma} V^{\gamma} V^{\beta} \Phi^{\delta}\right], = 0 \tag{59}$$

If one considers $\frac{d\tau}{ds} \neq 0$, the two metrics can be related to each other by means of a quasimetric one [31].

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - h_{\mu\nu} + \alpha_g (g_{\mu\nu} - U_{\mu} U_{\nu}) + \alpha_h (h_{\mu\nu} - V_{\mu} V_{\nu}), \tag{60}$$

such that

$$L = \tilde{g}_{\alpha\beta} U^{\alpha} \frac{\tilde{D}\Psi^{\beta}}{\tilde{D}S}, \tag{61}$$

$$\tilde{\Gamma}^{\alpha}_{\beta\sigma} = \frac{1}{2} \tilde{g}^{\alpha\delta} (\tilde{g}_{\sigma\delta,\beta} + \tilde{g}_{\delta\beta,\sigma} - \tilde{g}_{\beta\sigma,\delta})$$

and its corresponding Lagrangian:

$$L = \tilde{g}_{\mu\nu} U^{\mu} (\frac{d\Psi^{\nu}}{dS} + \tilde{\Gamma}^{\nu}_{\rho\delta} \Psi^{\rho} U^{\delta}) \tag{62}$$

Thus, equation of its path equation can be obtained by taking the variation respect to ψ^{μ} to obtain:

$$\frac{\tilde{D} U^{\alpha}}{\tilde{D} S^{2}} = 0 \tag{63}$$

while taking the variation with respect to U^{μ} to obtain its corresponding path deviation equation:

$$\frac{\tilde{D}^{2} \Psi^{\alpha}}{\tilde{D} S^{2}} = \tilde{R}^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} \tag{64}$$

where

$$\tilde{R}^{\alpha}_{.\mu\nu\rho} = \tilde{\Gamma}^{\alpha}_{\mu\rho,\nu} - \tilde{\Gamma}^{\alpha}_{\mu\nu,\rho} + \tilde{\Gamma}^{\sigma}_{\mu\rho} \tilde{\Gamma}^{\alpha}_{\sigma\rho} - \tilde{\Gamma}^{\sigma}_{\mu\rho} \tilde{\Gamma}^{\alpha}_{\sigma\rho}$$

Consequently, the following path and path deviation of charged and spinning objects of constant spinning tensor are explained as follows

$$L = \tilde{g}_{\alpha\beta} U^{\alpha} \frac{\tilde{D} \Psi^{\beta}}{\tilde{D} S} + \frac{e}{m} F_{\alpha\beta} U^{\alpha} \Psi^{\beta} \tag{65}$$

to give

$$\frac{\tilde{D} U^{\alpha}}{\tilde{D} S} = \frac{e}{m} F^{\mu}_{.\nu} U^{\nu} \tag{66}$$

and its corresponding deviation equation becomes:

$$\frac{\tilde{D}^{2} \Psi^{\alpha}}{\tilde{D} S^{2}} = \tilde{R}^{\alpha}_{.\mu\nu\rho} U^{\mu} U^{\nu} \Psi^{\rho} + \frac{e}{m} (F^{\alpha.\nu} \frac{\tilde{D} \Psi^{\nu}}{\tilde{D} s} + F^{\alpha}_{.\nu||\rho} U^{\nu} \Psi^{\rho}). \tag{67}$$

where $||$ represents the covariant derivative with respect to affine connection $\Gamma^{\alpha}_{\beta\sigma}$. Also, the generalized path and path deviation equations for spinning objects are obtained from the following Lagrangian:

$$L = \tilde{g}_{\mu\nu} \frac{\tilde{D} \Psi^{\alpha}}{\tilde{D} S} + \frac{1}{2m} \tilde{R}_{\alpha\mu\nu\rho} S^{\nu\rho} U^{\mu} \Psi^{\alpha} \tag{68}$$

By taking the variation with respect Ψ^{α} the to obtain its corresponding path equation:

$$\frac{\tilde{D} U^{\alpha}}{\tilde{D} S} = \frac{1}{2m} \tilde{R}^{\alpha}_{\beta\mu\nu} S^{\mu\nu} U^{\beta}, \tag{69}$$

and taking the variation with respect U^{α} to obtain its path deviation equation:

$$\frac{\tilde{D}^{2} \Psi^{\alpha}}{\tilde{D} S^{2}} = \tilde{R}^{\alpha}_{\beta\gamma\delta} U^{\gamma} U^{\beta} \Psi^{\delta} + \frac{1}{2m} (\tilde{R}^{\alpha}_{\beta\mu\nu} S^{\mu\nu} U^{\beta})_{||\rho} \Psi^{\rho} + \frac{1}{2m} \tilde{R}^{\alpha}_{\beta\mu\nu} S^{\mu\nu} U^{\beta} U_{\rho} \frac{\tilde{D} \Psi^{\rho}}{\tilde{D} S} \tag{70}$$

4 Discussion and Concluding Remarks

In this study, we have obtained the corresponding equations of path and path deviation equations for test particles, charged and spinning objects -constant spinning tensor- in different versions of Bimetric theories of gravity using a modified Bazanski Lagrangian. This type of study has imposed us to determine prior to its procedure some relevant path and path deviation for different objcts equations in Weyl geometry and MOND paradigm to be counted as an introductory step to visualize the different stages of path and path deviation equations that must be included before dealing with different bimetric theories of gravity. The study may give rise to search of a possible geometry able to express bimetric theory of gravity. It can be sought that Finslerian geometry is a good candidate to express bi-metric theory of gravity as an extension of Riemannian geometry [33]. In the mean time, path and path deviation equations using the Bazanski Lagrangian in Finsler geometry is in preparation [37]. Also, the above treatment of utilizing symmetric affine connection, can be extended into another version of bimetric theory of gravity following Einstein-Cartan geometry,as an extended approach of Drummond [38] which gives rise to different types of torsion and how does it propagate with respect to metric propagation due bimetric formalism. Finally, this work will enable us to examine, the stability of objects orbiting very strong gravitational field by solving the spin and spin deviation equations.

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