On the Vibrational Analysis for the Motion of a Rotating Cylinder

M. A. Bek, Tarek Amer, and Mohamed Abohamer

Abstract The main purpose of this work is to study the motion of 2-DOF of an auto-parametric dynamical system attached with a damped system. The governing equations of motion are gained utilizing Lagrange's equations in terms of the generalized coordinates. The method of multiple scales (MS) is used to obtain the solutions of the governing equations up to the third order of approximation. The primary external resonance simultaneously with the internal one are investigated to establish the solvability conditions and the modulation equations. The graphical representations of the time histories together with the amplitude and phases of the dynamical system are represented in some plots to describe the motion of the system at any instance. The stability of the solution has been made with use of *Mathematica.*

Keywords Vibration · Nonlinear dynamics · Stability

1 Introduction

Rotating cylinder over circular surface is widely appeared in many engineering applications. For example, electric motors, vibrating buildings, aviation, missiles, and locomotive engines are increasingly used as an example of it. As a rule,

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a rotating cylinder might be one of the main vibration's sources. Hence, it is very important to fully understand the vibration motion in order to offer better engineering solutions to reduce the vibration using a good design. Thus, various type of frequencies and mode shapes of such vibrating structures are significant in the design stage. Hence, it is very important to carefully understand where the resonance occurs to avoid structural familiar. For instant the proposed model is a good example of such systems. Discussion of such models may be found in [1–3]. The auto parametric resonance phenomena is observed as a cause of the coupling occurring in the equations of motion. Dynamical analysis of nonlinear vibrations of a mass of a cylinder shape which is rotated over circular body of radius R was presented in the paper. With the use of the multiple scale method the solution up to the third order is achieved [4, 5]. The system stability is investigated using Routh-Hurwitz criterion. Mathematica was the selected software to solve the algebra system and to obtain the results and graphically present it [6].

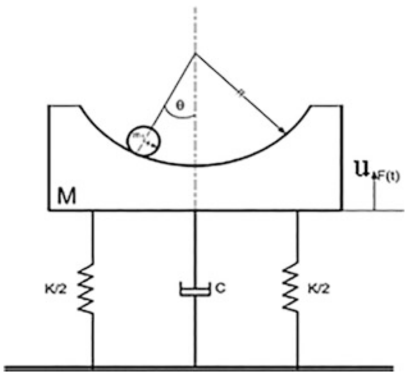
2 Dynamical Modeling

Let us consider the planar motion of a cylinder where its mass is m_1 and its radius is r over a circular surface of radius R and mass M. The system of the two masses is attached with the ground with an elastic spring of k spring stiffness, a damper of c damping coefficient, g earth's acceleration, and θ generalized co-ordinates admitted according to Fig. 1. The motion is considered under the influence of an external force $F(t)$ in the vertical direction.

Lagrange's equations were used to obtain the governing equation of motion for the corresponding system, the following

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) - \left(\frac{\partial L}{\partial u}\right) = Q_u, \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = Q_\theta \end{aligned} \tag{1}$$

Fig. 1 The dynamical model



Where Q_θ and Q_u represent the general forces, which have the forms

$$Q_u = f(t),\, Q_\theta = 0$$

The equations of motion can be written as

$$\begin{aligned} &\ddot{u} + \omega_1^2 u + m\left(\ddot{\theta}\ \sin\theta + \dot{\theta}^2\ \cos\theta\ \right) + 2cu\ \dot{u} = f(t) \\ &\ddot{\theta} + \left(\omega_2^2 + 2\ddot{u}\right)\sin\ \theta = 0, \end{aligned} \tag{2}$$

where

$$u = \tfrac{y-y_c}{R-r},\ \ m = \tfrac{m_1}{(m_1+M)},\ \ \omega_1^2 = \tfrac{k}{(m_1+M)}, \omega_2^2 = \tfrac{2g}{3(R-r)}, f(t) = \tfrac{F(t)}{[(R-r)(m_1+M)]} \tag{3}$$

The external force has the form $F(t) = F\cos\,(\Omega_1\, t)$.

The higher order of the trigonometric of functions $\cos\theta = 1 - \frac{\theta^2}{2}$ and $\sin\theta = \theta - \frac{\theta^3}{6}$ are admitted.

3 The Proposed Method

The amplitudes of all oscillations are theoretic to be of the order of a small parameter ε. This can be expressed as

$$\theta(t) = \varepsilon\ \phi\,(t;\varepsilon)\,, \qquad u(t) = \varepsilon\ x\,(t;\varepsilon)\,, \tag{4}$$

where $0 \prec \varepsilon \prec\ \prec 1$. We seek the asymptotic solutions ϕ and x in the form of power series of ε as

$$\begin{aligned} \phi &= \textstyle\sum_{k=1}^{3}\varepsilon^k\phi_k\,(\tau_0,\tau_1,\tau_2) + O\left(\varepsilon^4\right), \\ x &= \textstyle\sum_{k=1}^{3}\varepsilon^k x_k\,(\tau_0,\tau_1,\tau_2) + O\left(\varepsilon^4\right), \end{aligned} \tag{5}$$

where $\tau_n = \varepsilon^n t$; $(n = 0, 1, 2)$ are different time scales.

The derivatives in terms of the new time scales will be written in the following form

$$\begin{aligned} \frac{d}{d\tau} &= \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \varepsilon^2\frac{\partial}{\partial\tau_2}, \\ \frac{d^2}{d\tau^2} &= \frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon\frac{\partial^2}{\partial\tau_0\partial\tau_1} + \varepsilon^2\left(\frac{\partial^2}{\partial\tau_1^2} + 2\frac{\partial^2}{\partial\tau_0\partial\tau_2}\right) + O\left(\varepsilon^3\right). \end{aligned} \tag{6}$$

Terms of $O(\varepsilon^3)$ and higher order in Eq. (6) are neglected. Presumptuous that the amplitudes of generalized forces, the damping coefficients and the eccentricity to be small and have the form

$$f = \varepsilon^3 \ \tilde{f}. \tag{7}$$

Substituting expressions (4), (5), (6), and (7) into Eq. (2) and equating coefficients of like powers of ε in both sides, one obtains three groups of partial differential equations according to the asymptotic solutions (5). Therefore, one obtains the following system that consists six partial linear differential equations as follows.

Order (ε)

$$\frac{\partial^2 x_1}{\partial \tau_0^2} + \omega_1^2 x_1 = 0. \tag{8}$$

$$\frac{\partial^2 \phi_1}{\partial \tau_0^2} + \omega_2^2 \phi_1 = 0, \tag{9}$$

Order of (ε^2)

$$\frac{\partial^2 x_2}{\partial \tau_0^2} + \omega_1^2 x_2 = -m\left(\frac{\partial \phi_1}{\partial \tau_0}\right)^2 - 2\frac{\partial^2 x_1}{\partial \tau_0 \partial \tau_1} - m\ \phi_1 \frac{\partial^2 \phi_1}{\partial \tau_0^2}, \tag{10}$$

$$\frac{\partial^2 \phi_2}{\partial \tau_0^2} + \omega_2^2 \phi_2 = -2\frac{\partial^2 \phi_1}{\partial \tau_0 \tau_1} - \frac{2}{3}\phi_1\ \frac{\partial^2 x_1}{\partial \tau_0^2}, \tag{11}$$

Order of (ε^3)

$$\begin{aligned} \frac{\partial^2 x_3}{\partial \tau_0^2} + \omega_1^2 x_3 = \tilde{F}_1 \cos\ \Omega_1\ \tau_0 - \frac{\partial^2 x_1}{\partial \tau_1^2} - 2\tilde{C}\frac{\partial x_1}{\partial \tau_0} - 2\left(\frac{\partial^2 x_1}{\partial \tau_0 \partial \tau_2} + \frac{\partial^2 x_2}{\partial \tau_0 \partial \tau_1}\right) \\ - 2\ m\ \left[\frac{\partial \phi_1}{\partial \tau_0}\frac{\partial \phi_1}{\partial \tau_1} + \frac{\partial \phi_1}{\partial \tau_0}\frac{\partial \phi_2}{\partial \tau_0} + \phi_1 \frac{\partial^2 \phi_1}{\partial \tau_0 \partial \tau_1} + \phi_2 \frac{\partial^2 \phi_1}{\partial {\tau_0}^2} + \phi_1 \frac{\partial^2 \phi_2}{\partial {\tau_0}^2}\right], \end{aligned} \tag{12}$$

$$\begin{aligned} \frac{\partial^2 \phi_3}{\partial \tau_0^2} + \omega_2^2 \phi_3 = \frac{1}{6}\omega_2^2\ \phi_1^3 - \frac{\partial^2 \phi_1}{\partial \tau_1^2} - 2\left(\frac{\partial^2 \phi_1}{\partial \tau_0 \partial \tau_2} + \frac{\partial^2 \phi_2}{\partial \tau_0 \partial \tau_1}\right) \\ - \frac{4}{3}\ \phi_1 \frac{\partial^2 x_1}{\partial \tau_0 \partial \tau_1} - \frac{2}{3}\left(\phi_1 \frac{\partial^2 x_2}{\partial \tau_0^2} + \phi_2 \frac{\partial^2 x_1}{\partial \tau_0^2}\right), \end{aligned} \tag{13}$$

the previous partial differential equations system can be solved successively. In order to achieve this purpose, we start with the general solutions of Eqs. (8) and (9) in the form

$$x_1 = \mathrm{A}_1 e^{i\omega_1 \tau_0} + \overline{\mathrm{A}}_1 e^{-i\omega_1 \tau_0}, \tag{14}$$

$$\phi_1 = \mathrm{A}_2 e^{i\omega_2\tau_0} + \overline{\mathrm{A}}_2 e^{-i\omega_2\tau_0}, \tag{15}$$

where (A_i; $i = 1, 2, 3$) represent unknown complex functions of τ_1 and τ_2 while $\overline{\mathrm{A}}_i$ denotes to its complex conjugate.

Substitution of the solutions (14) and (15) into the equations of higher order (10) and (11) produce secular terms. In order to eliminate these terms, substituting (14) and (15) into (10) and (11) to obtain the required conditions for this purpose in the form

$$\frac{\partial \mathrm{A}_1}{\partial \tau_1} = 0, \quad \frac{\partial \mathrm{A}_2}{\partial \tau_2} = 0. \tag{16}$$

Consequently, the second order solutions become

$$x_2 = \frac{2\, m\omega_2^2 \mathrm{A}_2 \overline{\mathrm{A}}_2}{\omega_1^2} + \frac{e^{2i\omega_2\tau_0}\,(\omega_2 - 2\,i)\, m\,\omega_2\,{\mathrm{A}_2}^2}{\left(\omega_1^2 - 4\,\omega_2^2\right)} + CC, \tag{17}$$

$$\phi_2 = \frac{2\, e^{i(\omega_1+\omega_2)\tau_0}\omega_1 \mathrm{A}_1 \mathrm{A}_2}{3\,(\omega_1 + 2\omega_2)} + \frac{2\, e^{i(\omega_1-\omega_2)\tau_0}\omega_1 \mathrm{A}_1 \overline{\mathrm{A}}_2}{3\,(\omega_1 - 2\omega_2)} + CC, \tag{18}$$

where CC refers to the complex conjugates of the preceding terms.

Referring to the above procedure, the elimination of the secular terms required that the functions (A_i; $i = 1, 2$) depend upon the time scale τ_2 only.

The solutions for Eqs. (12) and (13) of the third order approximations can be obtained as previously in a similar way. The elimination of the secular terms in (12) and (13) demands the following conditions

$$-2\,i\,\tilde{C}\,\omega_1\,\mathrm{A}_1 - 2i\omega_1 \frac{\partial \mathrm{A}_1}{\partial \tau_2} - \frac{4\, m\omega_1^4 \mathrm{A}_1 \mathrm{A}_2 \overline{\mathrm{A}}_2}{3\left(\omega_1^2 - 4\,\omega_2^2\right)} = 0. \tag{19}$$

$$\begin{aligned} -2i\omega_2 \frac{\partial \mathrm{A}_2}{\partial \tau_2} & - \frac{4\,\omega_1^3\,\mathrm{A}_1 \mathrm{A}_2 \overline{\mathrm{A}}_1}{9\,(\omega_1 + 2\omega_2)} \\ & - \frac{\omega_2^2\left(3\,\omega_1^2 + 4\,\omega_2\,(-8\,i\,m + (-3 + 4m)\,\omega_2)\right)\mathrm{A}_2^2 \overline{\mathrm{A}}_2}{6\left(\omega_1^2 - 4\,\omega_2^2\right)} = 0. \end{aligned} \tag{20}$$

The solutions for Eqs. (12) and (13), after eliminating secular terms, take the form

$$\begin{aligned} x_3 = {} & \frac{\tilde{F}_1 e^{i\Omega_1\tau_0}}{2\left(\omega_1^2 - \Omega_1^2\right)} - \frac{2\,m\,\omega_1(\omega_1 + 2\omega_2)\mathrm{A}_1 \mathrm{A}_2^2 e^{i\tau_0(\omega_1 + 2\omega_2)}}{3\left[\omega_1^2 - (\omega_1 + 2\omega_2)^2\right]} \\ & + \frac{2\,m\omega_1(\omega_1 - 2\omega_2)\mathrm{A}_1 \overline{\mathrm{A}}_2^2 e^{i\tau_0(\omega_1 - 2\omega_2)}}{3\left[\omega_1^2 - (\omega_1 - 2\omega_2)^2\right]} + CC \end{aligned} \tag{21}$$

$$\phi_3 = -\frac{e^{3\,i\,\omega_2\tau_0}\left(\omega_1^2+4\,\omega_2(-8\,i\,m+(-1+4\,m)\omega_2)\right)\mathrm{A}_2^3}{48\left(\omega_1^2-4\,\omega_2^2\right)} - \frac{4\,e^{i(2\omega_1+\omega_2)\tau_0}\omega_1^3\mathrm{A}_1^2\mathrm{A}_2}{9(\omega_1+2\,\omega_2)\left[\omega_2^2-(2\omega_1+\omega_2)^2\right]} - \frac{4\,e^{i(2\omega_1-\omega_2)\tau_0}\omega_1^3\mathrm{A}_1^2\overline{\mathrm{A}}_2}{9(\omega_1-2\,\omega_2)\left[\omega_2^2-(2\omega_1-\omega_2)^2\right]} + CC, \tag{22}$$

The unknown functions (A_i; $i = 1, 2$) can be estimated from the system with the aid of the following initial conditions $\phi(0) = z_{01}, \quad \dot{\phi}(0) = z_{02,} \quad u(0) = z_{03}, \quad \dot{u}(0) = z_{04}, \quad .$

4 Vibrations and Resonance Conditions

Referring to the above procedure of the analytical solutions, the values of resonance parameters can be determined. The resonance cases can be attained if any of the polynomials in the denominators tends to zero and can be classified as

Primary external resonance, at $\Omega_1 = \omega_1$
Internal resonance occurred, if $\omega_1 = \omega_2$ are satisfied.

If any resonance case is satisfied, in particular the case of incidence internal resonance, we can predict that the dynamical behaviour of the system will be very complicated, the previous approximated solutions considered in the above section are valid if oscillations run away from resonances. If any one of the above listed conditions are satisfying; which indicates the need to adjust the used method.

5 External Resonances

5.1 Solvability Conditions

Here, we discuss the simultaneously occurring three primary external resonances case. Therefore, we consider that the combinations $\Omega_1 \approx \omega_1$ is satisfied which qualitatively describes the nearness of Ω_1 to ω_1. If we introduce the detuning parameters $(\sigma_i; i = 1, 2)$ as in the form

$$\begin{aligned} \Omega_1 &= \omega_1 + \sigma_1, \\ 2\omega_1 &= 2\omega_2 + \varepsilon\sigma_2\ . \end{aligned} \tag{23}$$

Then the effectiveness of resonance is reflected in the secular terms. The detuning parameters are considered a measure of distance of the vibrations from the strict resonance. Therefore, we express them in terms of the small parameter ε as

$$\sigma_i = \varepsilon\tilde{\sigma}_i; \quad i = 1, 2. \tag{24}$$

Substituting about the detuning parameters from (23) and (24) into Eq. (2) and focusing attention on the secular terms, then the solvability conditions are obtained as a result of the elimination of secular terms, in which that can be written as for the 2^{nd} order approximation

$$\frac{\partial A_1}{\partial \tau_1} = 0, \qquad \frac{\partial A_2}{\partial \tau_1} = 0, \tag{25}$$

-For the 3^{rd} approximation

$$\begin{aligned}
&\tfrac{1}{2}\tilde{F}_1\, e^{i\tau_1\tilde{\sigma}_1} - 2\,i\,\tilde{C}\,\omega_1\,\mathrm{A}_1 - 2i\omega_1\tfrac{\partial \mathrm{A}_1}{\partial \tau_2} - \tfrac{4\,m\omega_1^4\mathrm{A}_1\mathrm{A}_2\overline{\mathrm{A}}_2}{3\left(\omega_1^2-4\,\omega_2^2\right)} = 0,\\
&-\tfrac{4\,\omega_1^3\,\mathrm{A}_1^2\overline{\mathrm{A}}_2\,e^{i\tau_1\tilde{\sigma}_2}}{9(\omega_1-2\,\omega_2)} - 2i\omega_2\tfrac{\partial \mathrm{A}_2}{\partial \tau_2} - \tfrac{4\,\omega_1^3\,\mathrm{A}_1\mathrm{A}_2\overline{\mathrm{A}}_1}{9(\omega_1+2\omega_2)}\\
&-\tfrac{\omega_2^2\left(3\,\omega_1^2+4\,\omega_2(-8\,i\,m+(-3+4m)\omega_2)\right)\mathrm{A}_2^2\overline{\mathrm{A}}_2}{6\left(\omega_1^2-4\,\omega_2^2\right)} = 0.
\end{aligned} \tag{26}$$

6 Problem's Modulation Close to Resonances

Based on the above section, we can see that the solvability conditions of the considered model constitute a system consists of four nonlinear partial deferential equations in terms of unknown function A_i. It is worthwhile to notice from Eq. (25) that, the functions A_i depend upon the slow time scale τ_2 only. Therefore, we can express these functions in the polar notation as

$$A_i = \frac{\tilde{a}_i\,(\tau_2)}{2} e^{i\tilde{\psi}_i\tau_2}, \qquad a_i = \varepsilon\tilde{a}_i; \quad i = 1, 2. \tag{27}$$

Since A_i are independent functions of variable τ_0 andτ_1, then the first order derivative operator can be simplified to the form

$$\frac{\partial A_i}{\partial \tau} = \varepsilon^2\frac{\partial A_i}{\partial \tau_2}; \;\; (i = 1, 2)\,. \tag{28}$$

Bearing in mind the above formula (28), Eq. (26) turn into ordinary differential equations. In order to transform them into an autonomous ones, the modified phases can be introduced in the form

$$\begin{aligned}
\theta_1\,(\tau_1, \tau_2) &= \tau_1\;\tilde{\sigma}_1 - \psi_1\,(\tau_2)\,,\\
\theta_2\,(\tau_1, \tau_2) &= \tau_1\;\tilde{\sigma}_2 + 2\;\left[\psi_1\,(\tau_2) - \psi_2\,(\tau_2)\right].
\end{aligned} \tag{29}$$

Substituting (27), (28), and (29) into (26), separating the real and imaginary parts, one obtains directly the following system that consists of four ordinary differential equations from first order in terms of θ_1, θ_2, a_1, and a_2

$$
\begin{aligned}
a_1 \tfrac{d\theta_1}{d\tau} &= \tfrac{F_1}{2\omega_1}\cos\theta_1 + a_1\sigma_1 - \tfrac{m\, a_1 a_2^2\, \omega_1^3}{6\left(\omega_1^2 - 4\,\omega_2^2\right)}, \\
\tfrac{da_1}{d\tau} &= \tfrac{F_1}{2}\sin\theta_1 - \omega_1 \mu\, a_1, \\
a_2 \tfrac{d\theta_2}{d\tau} &= a_2\left(\sigma_2 + 2\sigma_1\right) - 2a_2\tfrac{d\theta_1}{d\tau} - \tfrac{\omega_1^3 a_1^3 a_2}{9\omega_2(\omega_1 + 2\omega_2)} \\
&\quad + \tfrac{a_2^3\left(3\,\omega_1^2\omega_2 + 4(-3+4m)\omega_2^3\right)}{24\left(\omega_1^2 - 4\,\omega_2^2\right)} - \tfrac{\omega_1^3 a_1^2 a_2\ \cos\theta_2}{9\omega_2(\omega_1 - 2\omega_2)}, \\
\tfrac{da_2}{d\tau} &= -\tfrac{2\, m\, \omega_2^2\, a_2^3}{3\left(\omega_1^2 - 4\,\omega_2^2\right)} - \tfrac{\omega_1^3 a_1^2\ \sin\theta_2}{18\omega_2(\omega_1 - 2\omega_2)}.
\end{aligned}
\tag{30}
$$

With a view to solve this system, we can transform the previous initial conditions according to the new variables as

$$a_i(0) = 0.004, \quad \theta_i(0) = 0, \, where\ i = 1, 2.$$

Equations (30) have the solutions a_1 and θ_1 that govern both of amplitudes and phases modulation in terms of the slow time scale when two investigated resonances occur with each other. Therefore, the solutions of these equations can be plotted as in Figs. 2, 3, 4, and 5 after taking into consideration the following values of parameters where $M = 25\,kg$, $k = 50$, $c = 0.01$, $\sigma 1 = 0.001$, $\sigma 2 = 0.003$. We calculated the time histories curves in Figs. 6 and 7 of the desired solutions φ and u up to the third approximations.

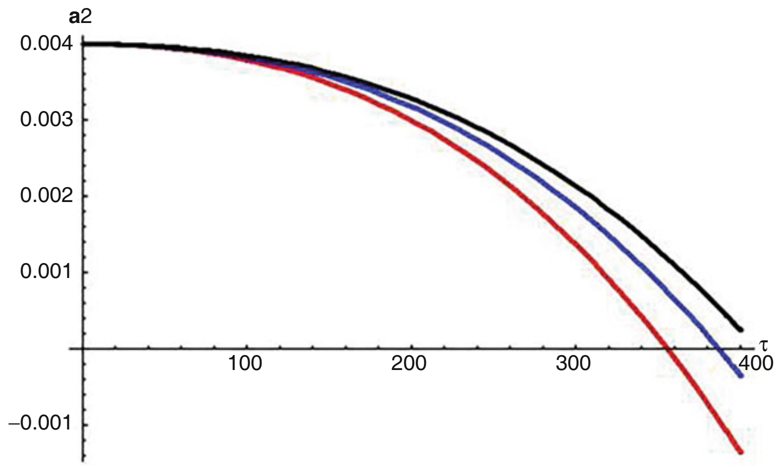


Fig. 2 Presentation of the variation of a_2

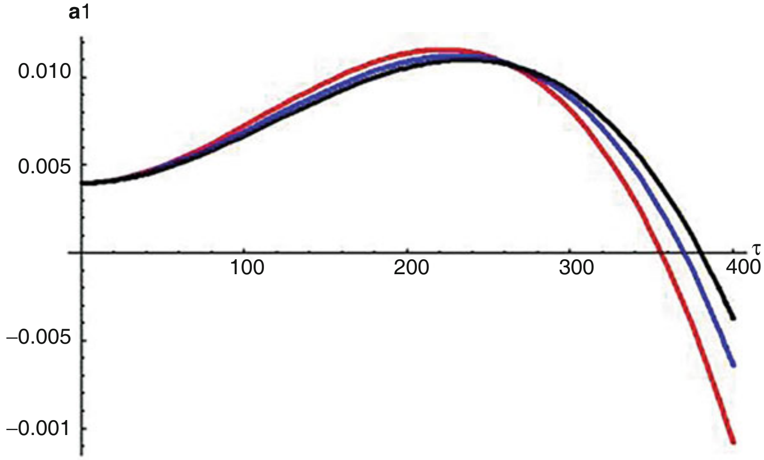


Fig. 3 Presentation of the variation of a_1

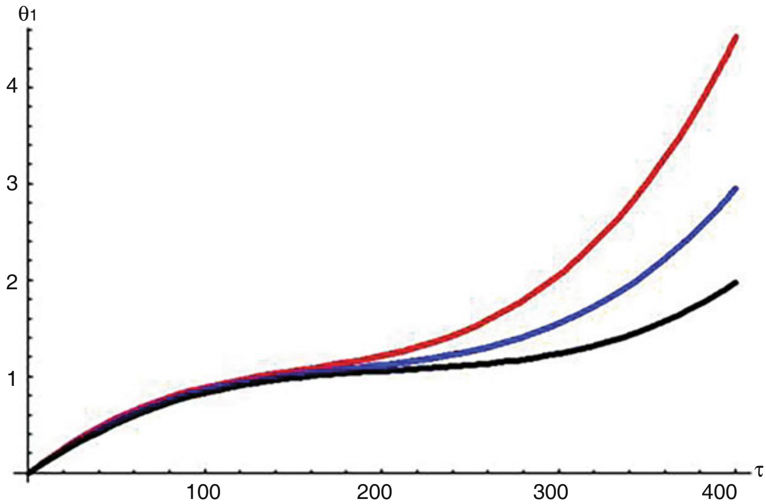


Fig. 4 Description of the variation of θ_1

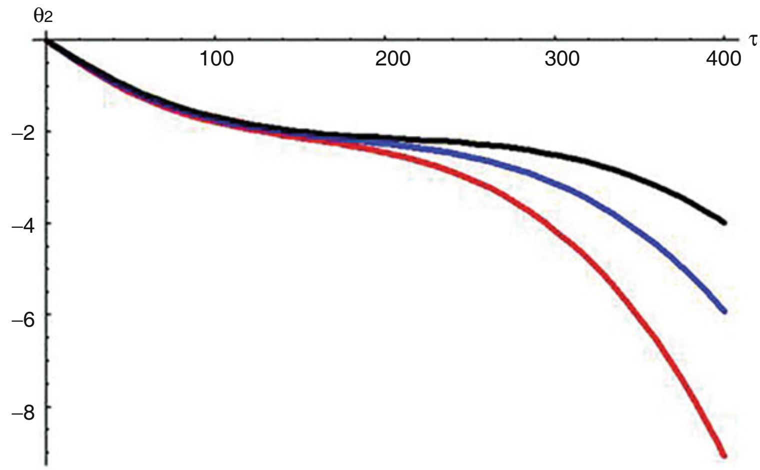


Fig. 5 Description of the variation of θ_2

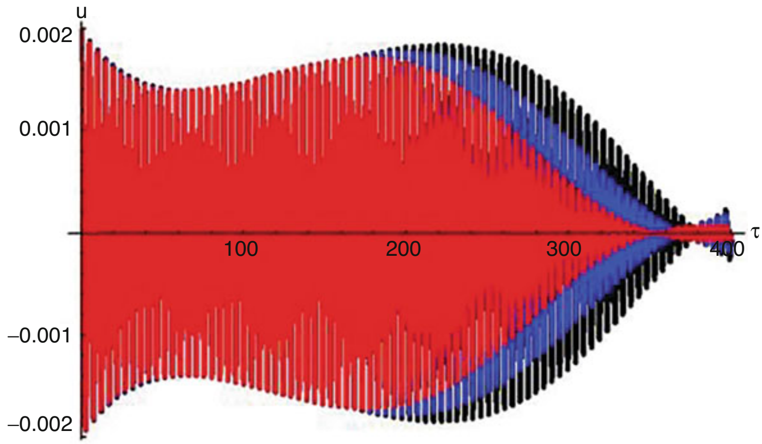


Fig. 6 Representation of the variation of u

7 Steady-State Solutions

The main objective of this section is to study the steady-state vibrations of the considered model. It is known that the steady state vibration arises if the behaviour of the transient processes disappears owing to the damping of the system. The amplitudes and modified phases of steady-state can be obtained from the Eq. (30), in order to explore such a case, let us assume that $\left(\frac{d\theta_i}{dt} = \frac{da_i}{dt} = 0; \;\; i = 1, 2, 3\right)$ equal

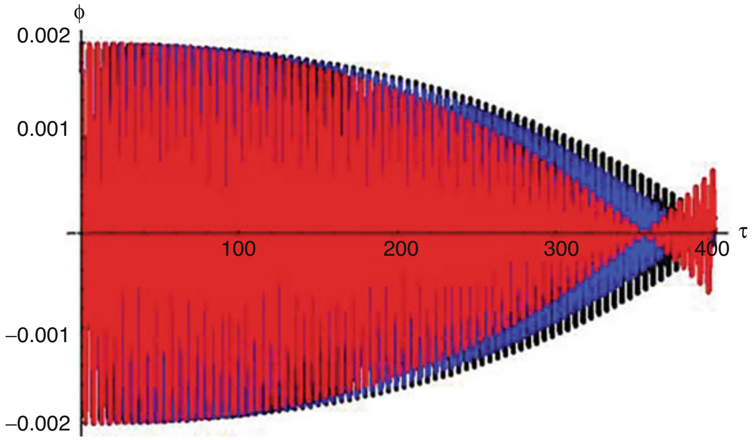


Fig. 7 Representation of the variation of φ

zero. Therefore, we have a system of algebraic equations in terms of the unknowns h_i and θ_i; $i = 1, 2$.

$$\begin{aligned}
&\tfrac{F_1}{2\omega_1}\cos\theta_1 + a_1\sigma_1 - \tfrac{m\, a_1 a_2^2\, \omega_1^3}{6(\omega_1^2-4\,\omega_2^2)} = 0,\\
&\tfrac{F_1}{2}\sin\theta_1 - \omega_1\mu\, a_1 = 0,\\
&a_2\,(\sigma_2 + 2\sigma_1) - \tfrac{\omega_1^3 a_1^2 a_2}{9\omega_2(\omega_1+2\omega_2)}\\
&\quad + \tfrac{a_2^3\,(3\,\omega_1^2\omega_2+4(-3+4m)\omega_2^3)}{24(\omega_1^2-4\,\omega_2^2)} - \tfrac{\omega_1^3 a_1^2 a_2\,\cos\theta_2}{9\omega_2(\omega_1-2\omega_2)} = 0,\\
&-\tfrac{2\,m\,\omega_2^2\,a_2^3}{3(\omega_1^2-4\,\omega_2^2)} - \tfrac{\omega_1^3 a_1^2\,\sin\theta_2}{18\omega_2(\omega_1-2\omega_2)} = 0.
\end{aligned} \tag{31}$$

Elimination of the modified phases θ_1 and θ_2 from Eq. (31), produces the relationships between both of the amplitudes and the frequency clarified by the detuning parameters

$$\begin{aligned}
F_1^2 &= \left[-\tfrac{m\,a_1 a_2^2\,\omega_1^4}{3(\omega_1^2-4\,\omega_2^2)} - 2a_1\omega_1\sigma_1\right]^2 + \omega_1^2\mu^2\,a_1^2,\\
\tfrac{\omega_1^6\,a_1^4}{81\,\omega_2^2(\omega_1-2\omega_2)^2} &= \left[(\sigma_2+2\sigma_1) - \tfrac{\omega_1^3 a_1^2}{9\omega_2(\omega_1+2\omega_2)} + \tfrac{a_2^3\,(3\,\omega_1^2\omega_2+4(-3+4m)\omega_2^3)}{24(\omega_1^2-4\,\omega_2^2)}\right]^2\\
&\quad + \left[\tfrac{4\,m\,\omega_2^2\,a_2^3}{3(\omega_1^2-4\,\omega_2^2)}\right]^2.
\end{aligned} \tag{32}$$

It is worthwhile to notice that Eq. (32) is considered as implicit nonlinear algebraic equation with respect to the variables a_1 and a_2.

8 Steady State Analysis

One of the important factors for the mentioned problem of the steady-state oscillations is to investigate their stability. For this task, we analyse the manner of the system in a region that is very close to the fixed points. To discuss the stability for the particular solution of the steady state, we introduce the substitutions into (30).

$$\begin{aligned} a_1 &= a_{10} + a_{11}, \\ a_2 &= a_{20} + a_{21}, \\ \theta_1 &= \theta_{10} + \theta_{11}, \\ \theta_2 &= \theta_{20} + \theta_{21}. \end{aligned} \tag{33}$$

Here a_{10}, θ_{10}, a_{20} and θ_{20} represent the solutions of (34) and a_{11}, θ_{11}, a_{21} and θ_{21} denote perturbations which are assumed to be very small, compared to the predecessors. Then the linearized equations take the form

$$\begin{aligned} a_{10}\frac{d\theta_{11}}{d\tau} &= -\frac{F_1}{2\omega_1}\theta_{11}\sin\theta_{10} + a_{11}\left[\sigma_1 - \frac{m\ \omega_1^3 a_{10}^2}{6\left(\omega_1^2 - 4\ \omega_2^2\right)}\right] - \frac{m\ \omega_1^3 a_{10} a_{20} a_{21}}{3\left(\omega_1^2 - 4\ \omega_2^2\right)}, \\ \frac{da_{11}}{d\tau} &= \frac{F}{2}\theta_{11}\cos\theta_{10} - \omega_1 \mu\ a_{11}, \\ a_{20}\frac{d\theta_{21}}{d\tau} &= a_{21}\left[\sigma_2 + 2\sigma_1 - \frac{\omega_1^3 a_{10}^2}{9\omega_2(\omega_1 + 2\omega_2)} + \frac{a_2^3\ \left(3\ \omega_1^2\omega_2 + 4(-3+4m)\omega_2^3\right)}{24\left(\omega_1^2 - 4\ \omega_2^2\right)}\right] \\ &\quad - 2a_{20}\frac{d\theta_{11}}{d\tau} - \frac{2\omega_1^3 a_{20} a_{10} a_{11}}{9\omega_2(\omega_1 + 2\omega_2)}, \\ \frac{da_{21}}{d\tau} &= -\frac{2m\ \omega_2^2\ a_{20}^2\ a_{21}}{\left(\omega_1^2 - 4\ \omega_2^2\right)} - \frac{\omega_1^3\ a_{20}\ \theta_{21}\cos\theta_{20}}{18\omega_2(\omega_1 - 2\ \omega_2)}. \end{aligned} \tag{34}$$

Take into consideration that the small perturbations a_{11}, θ_{11}, a_{21} and θ_{21} are unknown functions. Every solution is a linear combination of $k_i\, e^{\lambda\tau}$, where k_i $i = 1$, 2, 3, 4 are constants and λ is the eigenvalue corresponding to the unknown perturbation, counted from the real parts of the roots. In this analysis, if the steady-state solutions (fixed points) a_{10}, θ_{10}, a_{20} and θ_{20} are asymptotically stable, the real parts of the roots of the following characteristic equation

$$\lambda^4 + \Gamma_1\lambda^3 + \Gamma_2\lambda^2 + \Gamma_3\lambda + \Gamma_4 = 0. \tag{35}$$

of the set of (34), must be negative. Here Γ_1, Γ_2, Γ_3 and Γ_4 take the form

$$
\begin{aligned}
\Gamma_1 &= \tfrac{F\ \sin\theta_{10}}{2a_{10}\omega_1} + \tfrac{\mu\ \omega_1^3+2\left(m\ a_{20}^2-2\mu\ \omega_1\right)\ \omega_2^2}{\omega_1^2-4\ \omega_2^2},\\
\Gamma_2 &= \tfrac{1}{1296\ a_{10}a_{20}\ \omega_1\omega_2^2(\omega_1-2\omega_2)(\omega_1+2\omega_2)}\Big[108\ a_{20}\ \ (\omega_1-2\omega_2)\,\omega_2^2\big(24m\mu\ a_{10}a_{20}^2\omega_1^2\omega_2^2\\
&\quad +6\ F\sin\theta_{10}\left(\mu\omega_1^3+2\left(ma_{20}^2-2\mu\omega_1\right)\omega_2^2\right)\\
&\quad -F\ \omega_1\cos\theta_{10}\left(ma_{20}^2\omega_1^3-6\ \sigma_1\left(\omega_1^2-4\ \omega_2^2\right)\right)\big)\\
&\quad +a_{10}^2\ \omega_1^4\cos\theta_{10}\left(-8a_{10}^2\ \omega_1^3\left(\omega_1-2\omega_2\right)+3\omega_2\right)\\
&\quad \times\Big(48\sigma_1\left(\omega_1^2-4\ \omega_2^2\right)+24\sigma_1\left(\omega_1^2-4\ \omega_2^2\right)\\
&\quad +a_{20}^2\left(-16m\omega_1^3+9\omega_1^2\omega_2+12\left(-3+4m\right)\omega_2^3\right)\,))\Big],\\
\Gamma_3 &= \tfrac{1}{2592\ a_{10}a_{20}\omega_2^2\left(\omega_1^2-4\ \omega_2^2\right)^2}\Big[a_{10}\omega_1^2\ \cos\theta_{20}\left(\omega_1+2\omega_2\right)\big(-8a_{10}^2\omega_1^4\Big(F\ \ \sin\theta_{10}\\
&\quad +2\mu a_{10}\omega_1^2\ \big)+8\omega_1^2\big(\ F\ \ \sin\theta_{10}\left(18\sigma_1+9\sigma_2+2a_{10}^2\ \omega_1\right)+2\mu a_{10}\omega_1^2\\
&\quad \left(18\sigma_1+9\sigma_2+2\left(a_{10}^2-3ma_{20}^2\right)\omega_1\right)\Big)\omega_2+27a_{20}^2\omega_1^2\left(F\ \ \sin\theta_{10}+2\mu a_{10}\omega_1^2\right)\omega_2^2\\
&\quad -288\left(2\sigma_1+\sigma_2\right)\ \left(F\ \ \sin\theta_{10}+2\mu a_{10}\omega_1^2\right)\omega_2^3\\
&\quad +36\left(-3+4m\right)a_{20}^2\left(F\ \ \sin\theta_{10}+2\mu a_{10}\omega_1^2\right)\omega_2^4\\
&\quad -432\ F\ ma_{20}^2\ \omega_2^4\left(-6\mu\sin\theta_{10}\left(\omega_1^2-4\ \omega_2^2\right)\right.\\
&\quad \left.+\cos\theta_{10}\left(ma_{20}^2\ \omega_1^3-6\sigma_1\left(\omega_1^2-4\ \omega_2^2\right)\right)\right)\Big],\\
\Gamma_4 &= \tfrac{F\cos\theta_{20}\ \omega_1^3}{5184\ a_{20}\omega_2^2(\omega_1-2\omega_2)^3(\omega_1+2\omega_2)^3}\left[-3\omega_2(\,8\left(2\sigma_1+\sigma_2\right)\omega_1^2+3a_{20}^2\omega_1^2\omega_2\right.\\
&\quad -32\left(2\sigma_1+\sigma_2\right)\omega_2^2+4\left(-3+4m\right)a_{20}^2\omega_2^3\,)(\,-6\mu\sin\theta_{10}\left(\omega_1^2-4\ \omega_2^2\right)\\
&\quad +\cos\theta_{10}\left(ma_{10}^2\omega_1^3-6\sigma_1\left(\omega_1^2-4\ \omega_2^2\right)\right))\\
&\quad -8a_{10}^2\omega_1^3\left(\omega_1-2\omega_2\right)\left(2\mu\sin\theta_{10}\left(\omega_1^2-4\ \omega_2^2\right)\right.\\
&\quad \left.\left.+\cos\theta_{10}\left(ma_{10}^2\omega_1^3-6\sigma_1\left(\omega_1^2-4\ \omega_2^2\right)\right)\,)\right]\,.
\end{aligned}
\tag{36}
$$

However, according to the Routh-Hurwitz criterion, the fundamental conditions of the stability for the particular steady-state solutions will be

$$
\begin{aligned}
&\Gamma_1 > 0,\\
&\Gamma_3\left(\Gamma_1\Gamma_2-\Gamma_3\right)-\Gamma_4\Gamma_1^2 > 0,\\
&\Gamma_1\Gamma_2-\Gamma_3 > 0,\\
&\Gamma_4 > 0.
\end{aligned}
\tag{37}
$$

The stability of the system amplitudes are varying with different spring stiffness value. As presented in Figs. 8 and 9 the dashed line (on the lift side) represent the unstable region. Where, the solid line represents the stable region. The solid red line represents the stable region of a_1 and the blue color represents the stable region of a_2.

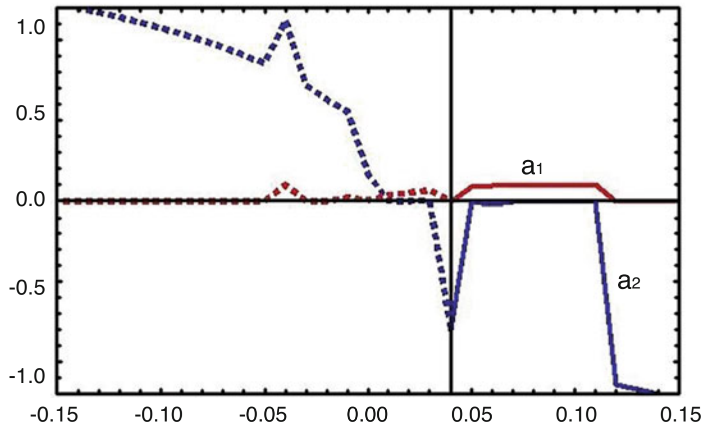


Fig. 8 stability for a_1 *and* a_2 *with* σ_1 *at* $k = 2000$

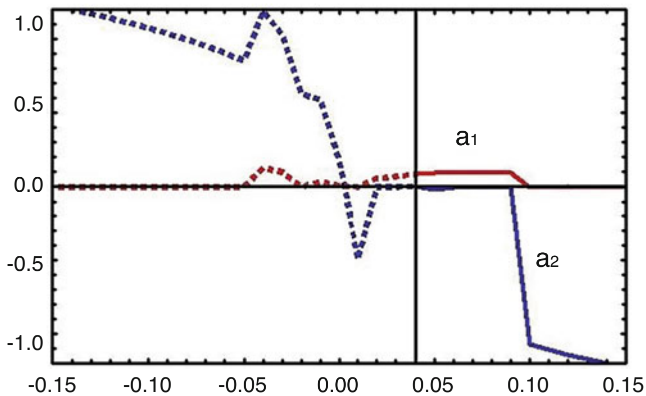


Fig. 9 stability for a_1 *and* a_2 *with* σ_1 *at* $k = 1500$

9 Conclusions

This work outlines the vibrating motion of a cylinder over circular under the influence of an exciting force. Lagrange's principles were utilized obtain the governing equation of the system's motion taking into account the presence of external forces acting on the vertical direction. The multiple Scale technique is used to obtain the asymptotic solutions up to third order and to gain the modulation equations in frame work of the solvability conditions. The various resonance cases, primary external resonance and internal one, are studied. The system stability is checked according to Routh-Hurwitz criterion condition. The graphical representations of time history of motion, resonance cases are presented through some plots to highlight the effectiveness of different physical parameters on the motion. The importance of this work is due to its direct applications in the fields of engineering machines which needs insight investigation in order to reduce the system vibrations.

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