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Chebyshev Pseudospectral Approximation for Solving Higher-Order Boundary Value Problems

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Abstract: In this paper, a Chebyshev Pseudospectral differentiation matrix is used to construct an approximate solution for higher-order boundary value problems. The algorithm developed approximates the numerical solution and their higher-order derivatives of sixth-order and fourth-order boundary value problems. The numerical results are compared with both the exact solution and the results of other methods. It is demonstrated that our algorithm is of high precision and efficiency.

Keywords: Chebyshev Pseudospectral differentiation matrix, higher-order boundary value problems.

1 Introduction

Many real life phenomena involve higher-order boundary value problems [1,3] based on ordinary differential equations. For example; the free vibrations analysis of beam structures is governed by a fourth-order ordinary differential equation. The vibration behavior of a ring structures is governed by a sixth-order ordinary differential equation. Moreover, when considering the instability setting in an infinite horizontal layer of fluid, which is heated from below and is subject to the action of rotation, we model the instability as ordinary convection by sixth-order ordinary differential equation. Such higher-order boundary value problems need to be solved numerically for the analytic solution sometimes can not be easily determined. Although, there have been many numerical solution approaches for solving ordinary differential equations in the literature, very few of these approaches are effective in solving such problems particularly when they are non-linear. This paper reports a fast and an accurate numerical scheme for solving linear or non linear higher-order boundary value problems with two-point boundary conditions. The Chebyshev Pseudospectral differentiation matrices are used efficiently and accurately to find numerical solutions for the higher-order boundary value problems in the form:

$$
\begin{aligned}
&y^{(2m)}(x) + f(x)p(y) = g(x)\,, \quad x \in [a,b] \\
&y^{(i)}(a) = \alpha_i, y^{(i)}(b) = \beta_i\,, \qquad 0 \le i \le m
\end{aligned}
\tag{1.1}
$$

where α_i and β_i; $i = 0, 1$ are finite real constants, and the functions $f(x)$, and $g(x)$ are continuous in $[a, b]$. For some choices of $f(x)$ and $g(x)$ the analytic solution can not be obtained.

Many authors represent several studies to solve higher-order boundary value problems. Agarwal [2] presented the theorems stating the conditions for existence and uniqueness of solutions of sixth-order boundary value problems, while no numerical methods of solutions of sixth-order boundary value problems are contained therein. A substantial amount of research work has been directed for such boundary value problems. Several numerical techniques such as sinc-Galerkin method [10,14], modified decomposition method [1] and B-spline method [13] have been implemented to solve such problems numerically. Also, the authors in [3,15,16] solved the sixth-order boundary value problems by using spline functions. The author in [18] applied differential transformation method to construct semi numerical-analytic solution of linear sixth-order boundary value problems with two-point boundary value conditions. Usmani [17] developed the method of the solution of fourth-order boundary value problem, considering it to be the problem of bending a rectangular clamped beam of length *l* resting on an elastic foundation. In [11] the authors applied a fixed point theorem to obtain sufficient conditions for existence of triple positive solutions of the symmetric nonlinear fourth-order boundary value problem and they applied it on elastic beam problem with higher order derivatives. Lidstone boundary value problems have been given considerable attention in the literature, and various forms of the problems have been discussed.

Pseudospectral methods are one of the principal methods of discretization for the numerical solution of differential equations. The main advantage of these methods lies in their accuracy for a given number of unknowns. For smooth problems in simple geometries, they offer exponential rates of convergence. Chebyshev Pseudospectral methods are widely used in the numerical approximation of many types of differential equations. The basic idea of the Pseudospectral methods consists of replacing the solution y by means of truncated Chebyshev expansion, and to compute spatial derivatives of y by analytic differentiation of the series. The linear map $D^{(1)}$, which takes a vector of N function values $\{y(x_k)\}$ to a vector of n derivatives values $\{y'(x_k)\}$, is known as the Pseudospectral differentiation matrix where N is the order of the approximation.

In this paper, an effective numerical method will be represented to solve the fourth-order and sixth-order boundary value problems. We will apply the Pseudospectral differentiation matrices based on Chebyshev polynomials by using an explicit formula [9] for higher order derivatives of Chebyshev polynomials.

The rest of this paper is organized as follows. Section 2 gives an idea about Chebyshev polynomials in numerical analysis. Section 3 presents the Chebyshev Pseudospectral differentiation matrices and the elements of the differentiation matrices. Section 4 explains how to use the Chebyshev differentiation matrices to get an approximate solution of the fourth-order and sixth order boundary value problems at Gauss Lobatto points. Section 5 presents the round-off error analysis of fifth and sixth-order derivatives. Section 6 is devoted for numerical examples. Two examples which are governed by sixth-order are employed to illustrate the proposed method. Also Pseudospectral Chebyshev differentiation matrices are used in section 6 to find a numerical solution for one of the most important fourth-order boundary problems in material science which describes the linear elastic beam bending. Two examples are considered.

2 Preliminaries

The *Chebyshev Polynomials* (of the first kind) are defined by:

$$T_n(x) = \cos(n\cos^{-1}(x)), n = 0, 1, 2, \ldots .N \tag{2.1}$$

The *Chebyshev-Gauss-Lobatto (CGL)* points are:

$$x_k = \cos\left(\frac{k\pi}{N}\right),\ k = 0, 1, ..., N \tag{2.2}$$

are a popular choice of quadrature points. The CGL points are $(n-1)$ extreme points of $T_n(x)$ in the relation (2.1) in addition to the endpoints of the interval$[-1, 1]$. The fundamental recurrence relation:

$$T_n(x) \ = \ 2xT_{n-1}(x) - T_{n-2}(x), n = \ 2,\ 3,\ \ldots \tag{2.3}$$

together with the initial conditions:

$$T_0(x) = 1,\ T_1(x) = x.$$

Since the range [0,1] is quite often more convenient to use the range [-1,1], we sometimes map the independent variable x in [0,1] to the variable S in [-1,1] by transformation $x = 0.5\,(1+S)$ that leads to a shifted Chebyshev polynomial of degree *n* in x on [0,1] given by:

$$T_n^*(x) = T_n(2x-1) \tag{2.4}$$

The basic idea of Pseudospectral approximations method is to assume the unknown $y(x)$ of the differential equation can be approximated by sum of *N*+1 basis functions $\phi_n(x)$ as follows:

$$y_N(x) = \sum_{n=0}^{N} a_n \phi_n(x) \tag{2.5}$$

This series is substituted into the differential equation: *Ly=f(x)*, $-1 \leq x \leq 1$, where L is a differential operator and $f(x)$ is given function. The result is called *residual function* and defined by:

$$R(x, a_0, a_1, ..., a_N) = Ly_N - f$$

This is identically equal to zero for the exact solution. This function is coupled with a set of prescribed conditions to constitutive the boundary value problem [5].
In the next section, we interest with non-periodic problems in [-1,1], whenever the best choice of $\phi_n(x)$ is Chebyshev polynomials which are taken with their associated collocation points in the interval [-1,1] given by a set of unequally Lobatto spaced Gauss-points (2.2). The set of collocation points is related to the set of basis function as the nodes of quadrature formulae which are used in the computation of the spectral coefficients from the grid values.

3 Chebyshev Pseudospectral differentiation matrices

For the Chebyshev collocation points (2.2) the first Pseudospectral differentiation matrix can be computed analytically [4]. The entries of D are given by:

$$d_{i,j} = \begin{cases} \frac{2N^2+1}{6} & i = j = 0 \\ -\frac{2N^2+1}{6} & i = j = N \\ -\frac{x_j}{2(1-x_j^2)} & i = j \neq 0, N \\ \frac{\gamma_i}{\gamma_j} \frac{(-1)^{(i+j)}}{x_i - x_j} & i \neq j \end{cases} \tag{3.1}$$

where γ_j=1 except for γ_0=γ_N=2
To calculate the higher order differentiation matrices, we can use $D^{(m)} = (D)^m$but it does not work for every set of collocation points. Other formulae based on discrete Chebyshev approximations are given in [9] as follows:

$$d_{i,j}^m = \frac{2\theta_j}{N} \sum_{k=m}^{N} \sum_{\substack{n = 0 \\ (n+k-m) even}}^{k-m} \theta_k b_{n.k}^m T_k(x_j) T_n(x_i) \tag{3.2}$$

with

$$b_{n.k}^m = \frac{2^m k}{(m-1)! c_n} \frac{(s-n+m-1)!(s+m-1)!}{(s)!(s-n)!}$$

where 2s =*k* + *n* – *m*, $c_0 = 2, c_n = 1, n \geq 1, \theta_j = 1$, except for $\theta_0 = \theta_N = \frac{1}{2}$.
These matrix entries can be replaced by one of the following formulas. The first is:

$$d^m_{i,j} = \frac{2\theta_j}{N} \sum_{k=m}^{N} \sum_{\substack{n=0 \\ (n+k-m) even}}^{k-m} \theta_k b^m_{n.k} \cos\frac{kj\pi}{N} \cos\frac{ni\pi}{N}. \tag{3.3}$$

The second formula depends on the periodic properties of the cosine function is follows:

$$d^m_{i,j} = \frac{2\theta_j}{N} \sum_{k=m}^{N} \sum_{\substack{n=0 \\ (n+k-m) even}}^{k-m} \theta_k b^m_{n,k} (-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]} x_{kj-N[\frac{kj}{N}]} x_{ni-N[\frac{ni}{N}]}. \tag{3.4}$$

However, the following are the differentiation matrices for $m = 1, 2, ..., 6$:

$$d^1_{i,j} = \frac{4\theta_j}{N} \sum_{k=1}^{N} \sum_{\substack{n=0 \\ (n+k) odd}}^{k-1} \frac{\theta_k k}{c_n} (-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]} x_{kj-N[\frac{kj}{N}]} x_{ni-N[\frac{ni}{N}]}$$

$$d^2_{i,j} = \frac{2\theta_j}{N} \sum_{k=2}^{N} \sum_{\substack{n=0 \\ (n+k) even}}^{k-2} \frac{\theta_k kpq}{c_n} (-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]} x_{kj-N[\frac{kj}{N}]} x_{ni-N[\frac{ni}{N}]}$$

$$d^3_{i,j} = \frac{\theta_j}{N} \sum_{k=3}^{N} \sum_{\substack{n=0 \\ (n+k) odd}}^{k-3} \frac{\theta_k k}{2c_n} (p^2-1)(q^2-1)(-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]} x_{kj-N[\frac{kj}{N}]} x_{ni-N[\frac{ni}{N}]}$$

$$d^4_{i,j} = \frac{\theta_j}{N} \sum_{k=4}^{N} \sum_{\substack{n=0 \\ (n+k) even}}^{k-4} \frac{\theta_k kpq}{12c_n} (p^2-4)(q^2-4)$$
$$\times (-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]} x_{kj-N[\frac{kj}{N}]} x_{ni-N[\frac{ni}{N}]}$$

$$d^5_{i,j} = \frac{\theta_j}{N} \sum_{k=5}^{N} \sum_{\substack{n=0 \\ (n+k) odd}}^{k-5} \frac{\theta_k k}{96\, c_n} (p^2-9)(q^2-9)(p^2-1)(q^2-1)$$

$$\times(-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]}x_{kj-N[\frac{kj}{N}]}x_{ni-N[\frac{ni}{N}]}$$

$$d_{i,j}^{6}=\frac{\theta_j}{N}\sum_{k=6}^{N}\sum_{\substack{n=0\\(n+k)even}}^{k-6}\frac{\theta_k k}{960\,c_n}pq\,(p^2-4)(q^2-4)(p^2-16)(q^2-16)$$

$$\times(-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]}\,x_{kj-N[\frac{kj}{N}]}x_{ni-N[\frac{ni}{N}]}$$

4 Pseudospectral approximation for solving higher-order boundary value problems

In this section, the numerical solutions of $y(x)$ for the problem (1.1) at Gauss Lobatto points (2.2) are presented and it based on Chebyshev approximations.
Let:

$$\tilde{y}^{(m)}(x_i)=\sum_{j=0}^{N}d_{i,j}^{(m)}y(x_j)$$

Then the problem (1.1) can be approximated by applying Chebyshev differentiation matrix as follows:

$$\sum_{j=0}^{N}d_{i,j}^{m}y(x_j)+f(x_i)y(x_i)=g(x_i),i=0,...,N$$

So that, we have a system of linear equations:

$$\left(D^{(m)}+F\right)[y]=[g]\,.$$

Here F is a diagonal matrix with elements$f(x_i)$, and the matrices $[g]$ and $[y]$ are two vectors with elements $g(x_i)$ and$y(x_i)$, respectively. The boundaries of the problem (1.1) are approximated as follows:

$$\sum_{j=0}^{N}d_{i,j}^{m}y(a)=\alpha_0,\sum_{j=0}^{N}d_{i,j}^{m}y(b)=\beta_i$$

Certainly, we are concerned with estimating the error in the numerical solution of matrix equations. The error in the solution arises from many sources, such as the round-off error in the numerical solution. Condition number is a widely used matrix feature in many areas, such as in numerical analysis and linear algebra. In numerical analysis, the condition number is basically a measure of stability or sensitivity of a matrix (or the linear system it represents) to numerical operations. The condition number is essential for estimating the error in the solution and has been a major topic in BVPs theory. Before proceeding, if the condition number of $D^{(m)}$ is large so it is called *ill conditioned*, whereas, it is called *well conditioned* matrix for the small condition number. The numerical experimental provided the fact [7] that:

$$\kappa(D) < \kappa(D^6)$$

where $\kappa(D)$ and $\kappa(D^6)$ are the condition numbers of the differentiation matrices D and D^6, respectively. This fact leads to transform the higher-order derivative given in (1.1) into first order derivative system as follows:

$$\left.\begin{array}{l} y_m'(x) + f(x)y_1 = g(x) \\ y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{m-1}' = y_m \end{array}\right\} \tag{4.1}$$

This system can be approximated as follows:

$$\left.\begin{array}{l} \sum\limits_{j=0}^{N} d_{i,j}^1 y_6(x_j) + f(x_i)y_1(x_i) = g(x_i) \\ \sum\limits_{j=0}^{N} d_{i,j}^1 y_1(x_j) = y_2(x_i) \\ \sum\limits_{j=0}^{N} d_{i,j}^1 y_2(x_j) = y_3(x_i) \\ \vdots \\ \sum\limits_{j=0}^{N} d_{i,j}^1 y_{m-1}(x_j) = y_m(x_i) \end{array}\right\} \tag{4.2}$$

Then, the system (4.2) can be written in matrix form as follows:

$$\left.\begin{array}{l} D[y_6] + F[y_1] = [g] \\ D[y_1] = [y_2] \\ D[y_3] = [y_4] \\ \vdots \\ D[y_{m-1}] = [y_m] \end{array}\right\} \tag{4.3}$$

5 Round-off error analysis of higher-order derivatives

Many authors represented many modifications of differentiation matrices and evaluate the elements of *D* and represented different techniques to compute the elements of the differentiation matrix [4].

The matrix (3.1) is not skew matrix as opposed to the Fourier differentiation matrix. If the collocation derivative is computed by matrix vector multiplication, then the total number of operations is $2N^2$ while this matrix can be computed in N^2 operations by using Solomonoff's algorithm [4]. It is important to remark that for large *N* the direct implementation of (3.1) suffers from cancellation, causing large errors in the elements of the matrix *D*.

The authors in [9] presented the absolute error in the evaluation of d_{01} and they found that the error is of order $O(N^4\varepsilon)$, where ε is the machine precision.

In this section, we will discuss the effect of round-off error on the elements $d^m_{i,j}$ in (3.4), for *m*=5,6, and we will prove that the error bounds of $d^m_{i,j}$ are of order $O(N^{2m}\varepsilon)$. In finite precision arithmetic, we have x^*_k=x_k+δ where $\delta = \max_k\{|\delta_k|\}$; and δ_k denotes a small error, with $|\delta_k|$ approximately equal to machine precision ε, and x^*_k is the exact value while x_k is the computed value with unit round-off $\varepsilon = 2.22 \times 10^{-16}$.

The absolute errors of the quantities $x_k x_n$ are:

$$|x^*_k x^*_n - x_k x_n| = (\delta_k + \delta_n) - O\left(\frac{1}{N^2}\delta_k\right) - O\left(\frac{1}{N^2}\delta_n\right).$$

5.1 Error bounds for the fifth-order derivative $d^5_{i,j}$

Considering Eq. (3.4), *m*=5, the error on the elements $d^5_{i,j}$ is given by:

$$d^{5*}_{i,j} - d^5_{i,j} = \frac{\theta_j}{N}\sum_{k=5}^{N}\sum_{\substack{n=0\\(n+k)odd}}^{k-5}\frac{\theta_k k}{96\,c_n}(p^2-9)(q^2-9)(p^2-1)(q^2-1)(-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]}\times$$

$$\left((\delta_{kj-N[\frac{kj}{N}]} + \delta_{ni-N[\frac{ni}{N}]}) - O(\frac{1}{N^2}\delta_{kj-N[\frac{kj}{N}]}) - O(\frac{1}{N^2}\delta_{ni-N[\frac{ni}{N}]})\right)$$

$$\leq \frac{\theta_j}{N}(\delta - O(\frac{1}{N^2}\delta))\sum_{k=5}^{N}\sum_{\substack{n=0\\(n+k)odd}}^{K-5}\frac{\theta_k k}{96\,c_n}(p^2-9)(q^2-9)(p^2-1)(q^2-1)$$

$$\leq \theta_j(\delta - O(\frac{1}{N^2}\delta))(\frac{N^{10}}{1013760} + \frac{11N^8}{552960} + \frac{5N^6}{129024} + \frac{11N^4}{30720}) \leq O(N^{10}\delta).$$

5.2 Error bounds for the sixth-order derivative $d^6_{i,j}$

Considering Eq. (3.4), *m*=6, the error on the elements $d^6_{i,j}$ is given by:

$$d^{6*}_{i,j} - d^6_{i,j} = \frac{\theta_j}{N}\sum_{k=6}^{N}\sum_{\substack{n=0\\(n+k)even}}^{k-6}\frac{\theta_k k}{960\,c_n}pq(p^2-4)(q^2-4)(p^2-16)(q^2-16)(-1)^{[\frac{kj}{N}]+[\frac{ni}{N}]}\times$$

$$\left((\delta_{kj-N[\frac{kj}{N}]}+\delta_{ni-N[\frac{ni}{N}]}) - O(\frac{1}{N^2}\delta_{kj-N[\frac{kj}{N}]}) - O(\frac{1}{N^2}\delta_{ni-N[\frac{ni}{N}]})\right)$$

$$\leq \frac{\theta_j}{N}(\delta - O(\frac{1}{N^2}\delta))\sum_{k=6}^{N}\sum_{\substack{n=0\\(n+k)even}}^{k-6}\frac{\theta_k k}{960\,c_n}pq(p^2-4)(q^2-4)(p^2-16)(q^2-16)$$

$$\leq \theta_j(\delta - O(\frac{1}{N^2}\delta))\left(\frac{4N^{12}}{135135} - \frac{16N^{10}}{10395} + \frac{8N^8}{315} + \frac{32209N^6}{7560}\right) \leq O(N^{12}\delta)$$

From the above results and those results in [7], it is clear that the error in $d^m_{i,j}$ is of order $O(N^{2m}\delta)$ for $m = 1, 2, ..., 6$. However, the differentiation matrix in (3.1) of the fifth-order derivative grows like$N^{12}\delta$, and the sixth-order increase as$N^{14}\delta$.

6 Numerical examples

6.1 Chebyshev approximation for solving sixth-order boundary value problems

Example 1: Consider the following linear sixth-order boundary value problem:

$$\begin{array}{ll} y^{(6)}(x) + y(x) = 6\cos(x), \quad x \in [0,1] & \\ y(0) = 0, & y(1) = 0, \\ y^{(1)}(0) = -1, & y^{(1)}(1) = \sin(1), \\ y^{(2)}(0) = 2, & y^{(2)}(1) = 2\cos(1) \end{array}$$

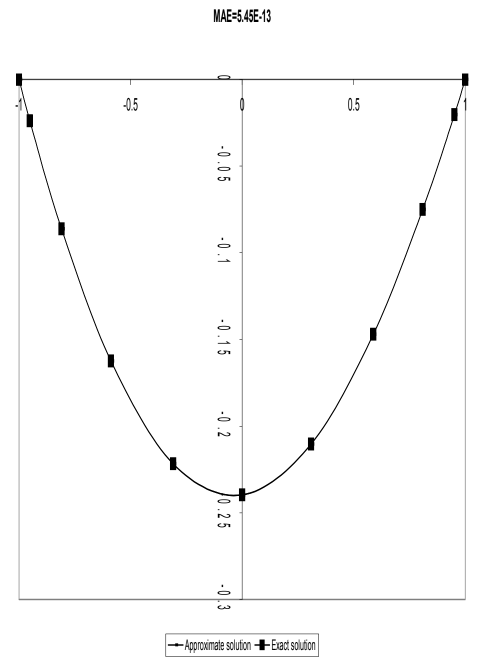
The exact solution is:

$$y(x) = (x-1)\sin(x)$$

By using the Chebyshev Pseudospectral approximation:

$\tilde{y}'_m(x_i) = \sum_{j=0}^{N} d^1_{i,j} y_m(x_j)$, for $m = 1, ..., 6$.

then the problem can be written as a system of first-order boundary value problem with $x \in [-1, 1]$



We use the method and obtain the results as shown in Fig.1.

Fig.1: The approximate solution and the exact solution for N=10 with MAE.

Table 1 shows the MAE compared with those determined in [16] while Table 2 shows the MAE in the derivatives associated with the problem.

Table (1): MAE for example 1

N	Presented method	Quintic spline [16]
8	7.2225E-10	1.8429E-6
16	9.0953E-12	1.3951E-7
32	6.1682E-10	9.4848E-9

Table (2): The MAE of the derivatives $y_N^{(1)}$, $y_N^{(2)}$, $y_N^{(3)}$, $y_N^{(4)}$, $y_N^{(5)}$ and $y_N^{(6)}$.

N	8	16	32
$y_N^{(1)}$	4.10E-09	1.40E-11	8.56E-10
$y_N^{(2)}$	6.55E-09	3.05E-11	2.02E-09
$y_N^{(3)}$	1.40E-09	1.75E-10	1.12E-08
$y_N^{(4)}$	3.95E-08	4.92E-10	3.30E-08
$y_N^{(5)}$	3.96E-08	4.82E-10	3.30E-08
$y_N^{(6)}$	1.65E-10	5.98E-13	9.07E-12

Example 2: Consider the following non linear sixth-order boundary value problem:

$$\begin{array}{ll} y^{(6)} + e^{-x}y^2 = e^{-x} + e^{-3x},\ x \in [0,1] & \\ y(0) = 1, & y(1) = \frac{1}{e} \\ y^{(1)}(0) = -1, & y^{(1)}(1) = -\frac{1}{e} \\ y^{(2)}(0) = 1\,, & y^{(2)}(1) = \frac{1}{e} \end{array}$$

The exact solution of the above system is given by**:**

$$y(x) = e^{-x}$$

Use the Chebyshev Pseudospectral approximation:

$\tilde{y}'_m(x_i) = \sum_{j=0}^{N} d^1_{i,j} y_m(x_j)$, for $m = 1, ..., 6$.

By applying the presented method we obtain the results as shown in Fig.2. Table 3 shows the MARE compared with those determined in [10] for *N*=10, while Table 4 shows MAE in the derivatives associated with the problem.

Table (3): The MARE of example 2 in *y* with N=10

Presented method	Sinc-Galerkin [10]
1.21E-11	0.1E-03

Table (4): MAE in the derivatives associated with example 2

N	y_N	$y_N^{(1)}$	$y_N^{(2)}$	$y_N^{(3)}$	$y_N^{(4)}$	$y_N^{(5)}$	$y_N^{(6)}$
8	1.0E-10	5.7E-10	5.2E-10	2.0E-09	6.2E-09	6.2E-09	9.8E-12
16	3.3E-10	4.1E-10	1.4E-09	6.0E-09	1.7E-08	1.7E-08	2.5E-12
32	1.4E-08	2.4E-08	4.3E-08	2.4E-07	7.0E-07	7.0E-07	9.8E-11

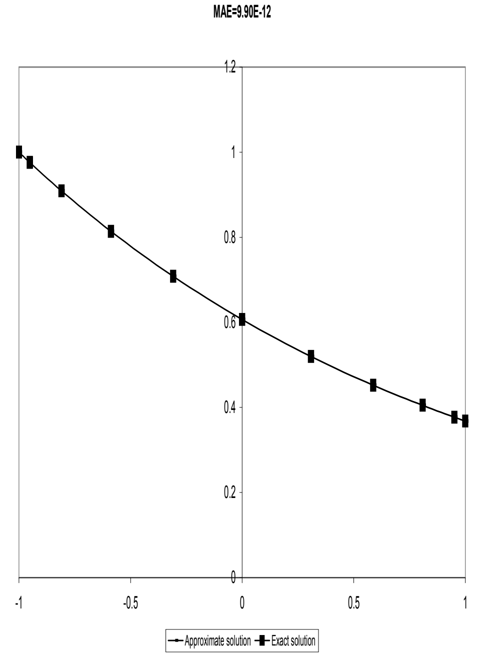
Fig.2: The approximate solution and the exact solution for *N*=10 with MAE

6.2 Chebyshev approximation for solving the beam bending boundary value problems

One of the most important equations in material science is the linear elastic beam equation. Finding efficient solutions for beam bending boundary vale problems is of great interest in engineering applications. These applications arise in building beam bridges, manufacturing of ships, submarines and planes. This section seeks an efficient solution for the problem of beam bending by using the Chebyshev Pseudospectral differentiation matrices. The following fourth-order boundary value problem represents a problem of rectangular beam bending of length l resting on an elastic foundation. The vertical deflection "ω" of the beam satisfies:

$$[L + (\frac{K}{S})]\omega = S^{-1}h(x) \tag{6.1}$$

where$L = \frac{d^4}{dx^4}$, S is the flexural rigidity of the beam and K is the spring constant of the elastic foundation and the load $h(x)$acts vertically downwards per unit length of the beam. Equation (6.1) is often referred to as the bending of an elastic beam under a variety of boundary conditions [17], and the boundary conditions are given by:



Clamped–clamped beam: $\omega(0) = \omega(l) = \omega^{(1)}(0) = \omega^{(1)}(l) = 0$,
Hinged–hinged beam: $\omega(0) = \omega(l) = \omega^{(2)}(0) = \omega^{(2)}(l) = 0,$
Clamped– hinged beam: $\omega(0) = \omega(l) = \omega^{(1)}(0) = \omega^{(2)}(l) = 0.$

Example 3: Consider the following boundary value problem which describes the model of the bending of a thin beam clamped at both ends:

$$y^{(4)} = \left(x^4 + 14x^3 + 49x^2 + 32x - 12\right) e^x,\ 0 \le x \le 1,$$
$$y(0) = y^{(1)}(0) = y(1) = y^{(1)}(1) = 0$$

The analytic solution is:

$$y(x) = x^2(1-x)^2 e^x$$

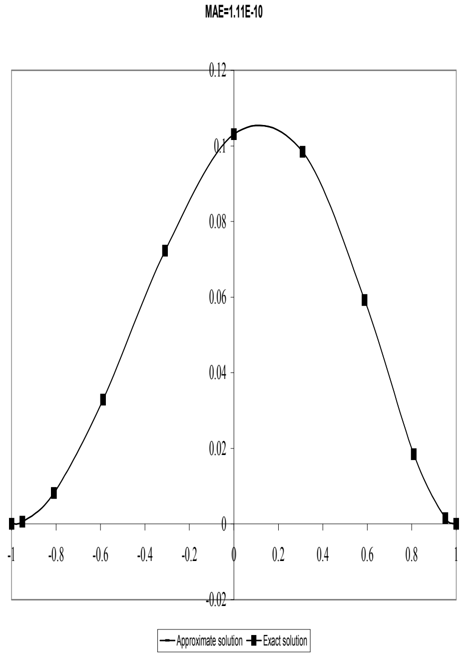
By mapping the variable *x* to [-1,1] and approximate the first-order derivatives $\tilde{y}'_m,\ m = 1, ..., 4$ as follows:

$$\tilde{y}'_m(x_i) = \sum_{j=0}^{N} d^1_{i,j} y_m(x_j)$$

Fig.3: The approximate solution and the exact solution for *N*=10 with MAE

Table (5): The MAE at *N*= 8,16,32,64,128.

N	8	16	32	64	128
MAE	9.3E-08	4.7E-13	3.1E-10	1.9E-11	8.73E-09



Example 4: Consider the following boundary value problem which describes the model of bending of a beam hinged from both sides:

$$y^{(4)}(x) = 1 - 4y(x), \quad -1 \le x \le 1,$$
$$y(\pm 1) = y^{(2)}(\pm 1) = 0$$

The exact solution is:

$$y(x) = \frac{1}{4}\left[1 - \frac{2(\sin 1 \sinh 1 \sin x \sinh x + \cos 1 \cosh 1 \cos x \cosh x)}{\cos 2 + \cosh 2}\right]$$

By applying the presented method the results at N=10 will be presented in Fig. 4

Fig. 4: The approximate solution and the exact solution for N=10 with MAE

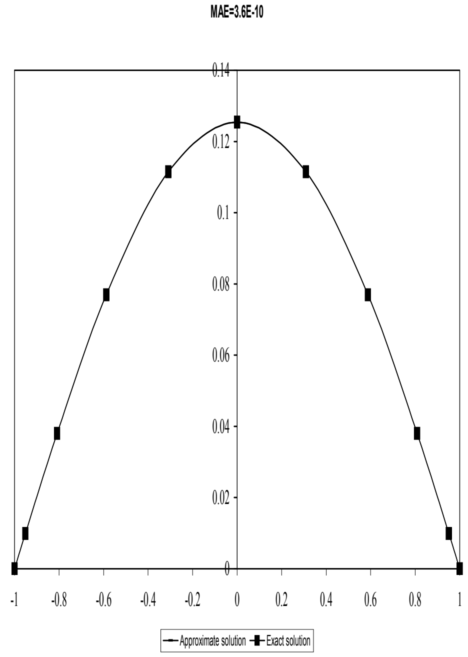
Table 6 represents the MAE at N=8, 16, 32, 64. Table 7 represents a comparison with the variational iteration methods [12] by using the MARE at N=8.

Table (6): The MAE at N= 8,16,32,64.

N	8	16	32	64
MAE	5.45E-08	9.43E-13	2.90E-11	5.64E-09

Table (7): The MARE at N=8

Presented method	Variational method [12]
1.01E-6	7.45E-03



7 Conclusion

The results of the previous section indicate that our procedure can be used to obtain accurate approximate solution for sixth-order and fourth-order boundary value problems. The method also approximates the derivatives of the solution up to the sixth-order. It has been observed that the relative errors in absolute, are better than those shown in [10] and the maximum absolute error are better than those shown in [1,3,12,14,15,16]. The method can be considered as an effective and reliable tool for solving sixth-order and fourth-order boundary value problems.

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