

# Path and path deviation equations for $p$ -branes

Research Article

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**Abstract:** Path and path deviation equations for neutral, charged, spinning and spinning charged test particles, using a modified Bazanski Lagrangian, are derived. We extend this approach to strings and branes. We show how the Bazanski Lagrangian for charged point particles and charged branes arises à la Kaluza-Klein from the Bazanski Lagrangian in 5-dimensions.

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## 1. Introduction

Theories of relativistic extended objects, called branes, have become one of the most promising branches of research in theoretical physics (for a review see [1]). They are a natural generalization of the concept of point particle. Branes occur at microscopic scales as a part of string/M-theory, or as fundamental objects of their own. On the macroscopic scale, a 3-brane living in a higher dimensional embedding space, can describe the entire universe. For a review of many aspects of embedding and the 'brane world' scenarios see [2].

A starting point of a brane theory is the Dirac-Nambu-Goto action, which gives the brane equations of motion – a generalization of the point particle geodesic equation.

As in the case of the point particle, the embedding space in which the brane moves can be curved. It is important to understand how a brane moves in such a classical background space, and how it deviates from the motion of a nearby brane. A pioneering work in that direction has been done by Roberts [3, 4].

Geodesic deviation equations for the point particle have been extensively studied in the literature. This is important, because from studying the free fall of two nearby objects we obtain the information on the curvature of space-time by using geodesic deviation equations. For instance, Ellis and Van Elst [5] used such equations for studying the structure of cosmological models. Recently, Wanas and Bakry [6] utilized geodesic deviation equations to examine the stability of some celestial objects. Also, there are some attempts by Roberts [7] who quantized geodesic and geodesic deviation equations. Bazanski [8, 9] discovered a very powerful method to obtain the geodesic and geodesic deviation equations from one single Lagrangian. Some authors, have applied this approach for examining

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path equations in different geometries than the Riemannian [10, 11], testing the effect of extra force on dynamical motion of spiral galaxies due to dark matter [12] and expressing the required paths of polyvectors as defined in curved Clifford spaces [13].

The aim of this work is to derive the equations of minimal surface and minimal surface deviation by using a generalized version of the Bazanski Lagrangian for  $p$ -branes. In section 2 we review how this works for the point particle ( $p = 1$ ), and explain how Bazanski Lagrangian [8, 9] can be extended to include not only neutral point particles but also charged, spinning and spinning charged particles. In section 3 we present the relevant equations of minimal surface and minimal surface deviation for  $p$ -branes, and the case of spinning and charged  $p$ -branes as well. In section 4 we show how the Bazanski Lagrangian for charged point particles and charged  $p$ -branes follow à la Kaluza-Klein from the Bazanski Lagrangian in 5-dimensions. Finally, we give some concluding remarks on how this work could be extended.

## 2. The Bazanski approach for the point particle

Geodesic and geodesic deviation equations for a relativistic point particle can be obtained simultaneously by using the Bazanski Lagrangian [8, 9]:

$$L = g_{\alpha\beta} u^\alpha \frac{D\psi^\beta}{Ds}. \quad (1)$$

Here  $u^\alpha \equiv dX^\alpha/ds$ , where  $X^\alpha$  are the particle's coordinates, and  $s$  the proper time, whereas  $\psi^\beta$  is the  $s$ -dependent deviation vector, associated with a one parameter family of geodesics  $X^\mu(s, \epsilon)$  according to [8, 9]

$$\psi^\mu = \epsilon \left. \frac{\partial X^\mu}{\partial \epsilon} \right|_{\epsilon=0}. \quad (2)$$

Performing the variation of the action

$$\mathcal{I}[X^\alpha, \psi^\alpha] = \int ds L \quad (3)$$

with respect to the deviation vector  $\psi^\rho$ , we obtain the geodesic equation:

$$\frac{du^\alpha}{ds} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} u^\mu u^\nu = 0. \quad (4)$$

If we vary the action (3) respect to  $X^\rho$ , then we obtain the geodesic deviation equation:

$$\frac{D^2 \psi^\alpha}{Ds^2} = R^\alpha{}_{\beta\gamma\delta} \psi^\gamma u^\beta u^\delta. \quad (5)$$

The above Lagrangian can be generalized [14] to include the coupling of a charged particle with the electromagnetic field:

$$L = g_{\alpha\beta} u^\alpha \frac{D\psi^\beta}{Ds} + \frac{e}{m} F_{\alpha\beta} u^\alpha \psi^\beta. \quad (6)$$

Then, instead of Eq. (4), we obtain

$$\frac{du^\alpha}{ds} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} u^\mu u^\nu = \frac{e}{m} F^\mu{}_\nu u^\nu. \quad (7)$$

On the other hand, if we consider the Lagrangian

$$L = g_{\alpha\beta} u^\alpha \frac{D\psi^\beta}{Ds} + \frac{1}{2m} R_{\alpha\beta\gamma\sigma} u^\alpha \psi^\beta S^{\gamma\sigma}, \quad (8)$$

we obtain [14] the Papapetrou equation:

$$\frac{du^\alpha}{ds} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} u^\mu u^\nu = \frac{1}{2m} R^\alpha{}_{\mu\nu\rho} S^{\rho\nu} u^\mu, \quad (9)$$

which, together with  $DS^{\mu\nu}/Ds = 0$ , is the equation of motion for a relativistic top moving in a gravitational field background. The latter equation for  $S^{\mu\nu}$  means that our object does not precess. In the Lagrangian (8) we consider the spin tensor  $S^{\mu\nu}$  as a fixed quantity that is not varied. Later we shall consider an action in which also  $S^{\mu\nu}$  is a dynamical variable.

Dixon equation for rotating charged objects

$$\frac{du^\alpha}{ds} + \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} u^\mu u^\nu = \frac{e}{m} F^\mu{}_\nu u^\nu + \frac{1}{2m} R^\alpha{}_{\mu\nu\rho} S^{\rho\nu} u^\mu \quad (10)$$

follows from the Lagrangian that combines eqs. (6) and (8):

$$L = g_{\alpha\beta} u^\alpha \frac{D\psi^\beta}{Ds} + \frac{e}{m} F_{\alpha\beta} u^\alpha \psi^\beta + \frac{1}{2m} R^\alpha{}_{\mu\nu\rho} S^{\nu\rho} u^\mu. \quad (11)$$

Whilst the conventional Bazanski Lagrangian (1) gives geodesic and geodesic deviation equations, the trajectories of charged and spinning objects can be obtained from the modified Lagrangians (6), (8). This has led Kahlil [14] to suggest the following general Lagrangian:

$$L = g_{\alpha\beta} u^\alpha \frac{D\psi^\beta}{Ds} + f_\beta \psi^\beta, \quad (12)$$

where

$$f_\beta = a_1 F_{\alpha\beta} u^\beta + a_2 R_{\alpha\beta\gamma\delta} S^{\gamma\delta} u^\alpha.$$

Here  $a_1$  and  $a_2$  are parameters that may take the values  $\frac{e}{m}$  and  $\frac{1}{2m}$  to be adjusted with the original Lorentz force equation and the Papapetrou equation as well as the Dixon equation.

Let us now consider the deviation equations that correspond to Lagrangians (6), (8), and (11).

(i) Charged deviation equation:

$$\frac{D^2 \psi^\alpha}{Ds^2} = R^\alpha{}_{\mu\nu\rho} u^\mu u^\nu \psi^\rho + \frac{e}{m} \left( F^\alpha{}_\nu \frac{D\psi^\nu}{Ds} + F^\alpha{}_{\nu;\rho} u^\nu \psi^\rho \right). \quad (13)$$

(ii) Rotating deviation equations (without precession)

$$\begin{aligned} \frac{D^2 \psi^\alpha}{Ds^2} = & R^\alpha{}_{\mu\nu\rho} u^\mu u^\nu \psi^\rho + \frac{1}{2m} \left( R^\alpha{}_{\mu\nu\rho} S^{\nu\rho} \frac{D\psi^\mu}{Ds} \right. \\ & \left. + R^\alpha{}_{\mu\nu\lambda} S^{\nu\lambda}{}_{;\rho} u^\mu \psi^\rho + R^\alpha{}_{\mu\nu\lambda;\rho} S^{\nu\lambda} u^\mu \psi^\rho \right). \end{aligned} \quad (14)$$

(iii) Rotating charged deviation equations:

$$\begin{aligned} \frac{D^2 \psi^\alpha}{Ds^2} = & R^\alpha{}_{\mu\nu\rho} u^\mu u^\nu \psi^\rho + \frac{e}{m} \left( F^\alpha{}_\nu \frac{D\psi^\nu}{Ds} + F^\alpha{}_{\nu;\rho} u^\nu \psi^\rho \right) \\ & + \frac{1}{2m} \left( R^\alpha{}_{\mu\nu\rho} S^{\nu\rho} \frac{D\psi^\mu}{Ds} + R^\alpha{}_{\mu\nu\lambda} S^{\nu\lambda}{}_{;\rho} u^\mu \psi^\rho \right. \\ & \left. + R^\alpha{}_{\mu\nu\lambda;\rho} S^{\nu\lambda} u^\mu \psi^\rho \right). \end{aligned} \quad (15)$$

The above deviation equations (13), (14), and (15) can also be derived directly from the equations of motion (7), (9), and (10), respectively. For example, Nieto et al. [15] derived in such a manner the deviation equation (14) from equation (9) for two nearby tops.

Papapetrou [16] has formed an equation of a rotating object which is able to precess:

$$\frac{D}{Ds} \left( m u^\alpha + u_\beta \frac{DS^{\alpha\beta}}{Ds} \right) = \frac{1}{2} R^\alpha{}_{\mu\nu\rho} S^{\rho\nu} u^\mu. \quad (16)$$

Kahil has suggested the following Lagrangian [14]:

$$L = g_{\alpha\beta} \left( m u^\alpha + u_\beta \frac{DS^{\alpha\beta}}{Ds} \right) \frac{D\psi^\beta}{Ds} + \frac{1}{2} R_{\alpha\beta\gamma\delta} S^{\gamma\delta} u^\alpha \psi^\beta \quad (17)$$

which gives the Papapetrou equation (16), and the following path deviation equation:

$$\begin{aligned} \frac{D^2 \psi^\alpha}{Ds^2} = & \frac{1}{m} R^\alpha{}_{\mu\nu\rho} u^\mu \left( m u^\nu + u_\beta \frac{DS^{\nu\beta}}{Ds} \right) \psi^\rho \\ & - \frac{1}{m} g^{\alpha\sigma} g_{\nu\lambda} u_\beta \left( \frac{DS^{\lambda\beta}}{Ds} \right)_{;\sigma} \frac{D\psi^\nu}{Ds} \\ & + \frac{1}{2m} \left( R^\alpha{}_{\mu\nu\rho} S^{\nu\rho} \frac{D\psi^\mu}{Ds} + R^\alpha{}_{\mu\nu\lambda} S^{\mu\lambda}{}_{;\rho} u^\nu \psi^\rho \right. \\ & \left. + R^\alpha{}_{\mu\nu\lambda;\rho} S^{\nu\lambda} u^\mu \psi^\rho \right). \end{aligned} \quad (18)$$

Let us now consider the following action

$$\begin{aligned} I[X^\alpha, \psi^\alpha, S^{\alpha\beta}, \psi^{\alpha\beta}] = & \int ds \left[ g_{\alpha\beta} \left( m u^\alpha + u_\rho \frac{DS^{\alpha\rho}}{Ds} \right) \frac{D\psi^\beta}{Ds} + \frac{1}{2} R_{\alpha\beta\gamma\delta} S^{\gamma\delta} u^\alpha \psi^\beta \right. \\ & \left. + S_{\alpha\beta} \frac{D\psi^{\alpha\beta}}{Ds} - u^\rho \left( \frac{DS_{\alpha\rho}}{Ds} u_\beta - \frac{DS_{\beta\rho}}{Ds} u_\alpha \right) \psi^{\alpha\beta} \right], \end{aligned} \quad (19)$$

which generalizes the action that corresponds to the Lagrangian (17). The action (19), in addition to the particle position  $X^\alpha$ , and the deviation vector  $\psi^\alpha$ , is a functional of the spin tensor  $S^{\alpha\beta}$  and the spin deviation tensor  $\psi^{\alpha\beta}$ . The latter quantity is defined with respect to a 1-parameter family of spin tensors  $S^{\alpha\beta}(s, \epsilon)$  according to

$$\psi^{\alpha\beta} = \epsilon \left. \frac{\partial S^{\alpha\beta}}{\partial \epsilon} \right|_{\epsilon=0}. \quad (20)$$

The variation of the action (19) with respect to  $\psi^\alpha$  gives again the Papapetrou equation (16), whereas the variation with respect to  $\psi^{\alpha\beta}$  gives

$$\frac{DS_{\alpha\beta}}{Ds} + u^\rho \left( \frac{DS_{\alpha\rho}}{Ds} u_\beta - \frac{DS_{\beta\rho}}{Ds} u_\alpha \right) = 0, \quad (21)$$

which is the equation of motion for the spin derived by Papapetrou [16].

Introducing

$$p_\alpha = m u_\alpha + u^\beta \frac{DS_{\alpha\beta}}{Ds} \quad (22)$$

we can write Eq. (21) in the form

$$\frac{DS_{\alpha\beta}}{Ds} = -(p_\alpha u_\beta - p_\beta u_\alpha). \quad (23)$$

The variation of the action (19) with respect to  $X^\alpha$  and  $S^{\alpha\beta}$ , gives a coupled system of equations for the deviation vector  $\psi^\alpha$  and the deviation tensor  $\psi^{\alpha\beta}$ . But, if we use the spin equations of motion (21) in the action (19), then the terms containing  $\psi^{\alpha\beta}$  become  $S_{\alpha\beta} \frac{D\psi^{\alpha\beta}}{Ds} + \frac{DS_{\alpha\beta}}{Ds} \psi^{\alpha\beta}$ , which is equal to the total derivative  $\frac{D}{Ds} (S^{\alpha\beta} \psi^{\alpha\beta})$ . Therefore, the extra terms have no influence on the equations of motions for variables  $X^\mu$  and  $\psi^\alpha$ , and we remain with the system, described by the action (16)-(18).

### 3. The Bazanski approach for the brane

In this approach we are going to introduce the following Lagrangian as a counterpart of the Bazanski Lagrangian

for the geodesic and geodesic deviation equations of the branes in curved embedding space

$$L = \kappa \sqrt{f} f^{ab} g_{\mu\nu} \partial_a X^\mu D_b \Psi^\nu. \quad (24)$$

Here  $\kappa$  is the brane tension,  $f^{ab}$  is the inverse of the induced metric on the brane,  $f_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$ ,  $f \equiv \sigma \det f_{ab}$ , where  $\sigma$  is plus or minus sign, so that  $f$  is positive regardless of the signature of  $f_{ab}$ , and  $\partial_a X^\mu$  is the vector velocity on the brane. The brane sweeps a worldsheet which is parametrized by coordinates  $\xi^a$ , its embedding functions being  $X^\mu(\xi^a)$ , and  $\partial_a \equiv \partial/\partial \xi^a$ .  $\Psi^\nu$  is the deviation vector field, associated with a one parameter family of minimal worldsheets  $X^\mu(\xi^a, \epsilon)$  according to

$$\Psi^\nu = \epsilon \left. \frac{\partial X^\mu}{\partial \epsilon} \right|_{\epsilon=0}. \quad (25)$$

A Lagrangian, similar to (24), but with the ordinary derivative  $\partial_b \Psi^\mu$ , instead of the covariant derivative  $D_b \Psi^\mu = \partial_b \Psi^\mu + \Gamma_{\rho\sigma}^\mu \Psi_\sigma \partial_a X^\rho$ , is considered in ref. [4].

Applying the variation with respect to the *deviation vector*  $\Psi^\nu$ , the Euler-Lagrange equation becomes

$$\frac{1}{\sqrt{f}} \partial_c (\sqrt{f} f^{ac} \partial_a X^\mu) + f^{ab} \Gamma_{\alpha\beta}^\mu \partial_a X^\alpha \partial_b X_\beta = 0, \quad (26)$$

which can be written more compactly as

$$D_c (f^{ac} \partial_c X^\mu) = 0, \quad (27)$$

where  $D_c \equiv D/D\xi^a$  is analogous to  $D/Ds$ , and denotes the covariant derivative with respect to the embedding metric  $g_{\mu\nu}$  and the world sheet metric  $f_{ab}$  (see Eq. (35)). Let us now derive the corresponding Euler-Lagrange equation, obtained by varying the action (24) with respect to  $X^\mu$ :

$$\partial_c \frac{\partial L}{\partial \partial_c X^\alpha} - \frac{\partial L}{\partial X^\alpha} = 0. \quad (28)$$

Using

$$\frac{\partial}{\partial X^\alpha} (\sqrt{f} f^{ab}) = \frac{1}{2} \sqrt{f} (f^{ef} f^{ab} - f^{ae} f^{bf} - f^{af} f^{be}) \partial_e X^\rho \partial_f X^\sigma g_{\rho\sigma, \alpha}, \quad (29)$$

$$\frac{\partial (\sqrt{f} f^{ab})}{\partial \partial_c X^\alpha} = \sqrt{f} (f^{cf} f^{ab} - f^{ac} f^{bf} - f^{af} f^{bc}) \partial_f X^\sigma g_{\alpha\sigma}, \quad (30)$$

and

$$\partial_c \Gamma_{\alpha\sigma}^\nu = \partial_\rho \Gamma_{\alpha\sigma}^\nu \partial_c X^\rho \quad (31)$$

we obtain

$$\begin{aligned} & D_c D^c \psi_\alpha + R_{\alpha\sigma\rho}^\nu g_{\mu\nu} \partial^c X^\mu \partial_c X^\rho \psi^\sigma \\ & + (1/\sqrt{f}) \partial_c \left[ \sqrt{f} (f^{cf} f^{ab} - f^{ac} f^{bf} - f^{af} f^{bc}) \right. \\ & \left. \partial_f X^\sigma g_{\alpha\sigma} \partial_a X^\mu D_b \psi^\nu g_{\mu\nu} \right] - \frac{1}{2} (f^{ef} f^{ab} - f^{ae} f^{bf} - f^{af} f^{be}) \\ & \partial_e X^\rho \partial_f X^\sigma g_{\rho\sigma, \alpha} \partial_a X^\mu D_b \psi^\nu g_{\mu\nu} = 0, \end{aligned} \quad (32)$$

where

$$\sqrt{f} D_c D^c \psi_\alpha = \partial_c (\sqrt{f} D^c \psi_\alpha) - \sqrt{f} \Gamma_{\rho\alpha}^\lambda D^c \psi_\lambda \partial_c X^\rho. \quad (33)$$

Eq.(32) is the equation of motion for the deviation vector. We can rewrite it in a more compact form by taking into account

$$\partial_c \sqrt{f} = \sqrt{f} \Gamma_{cd}^d, \quad \partial_c f^{ab} = -f^{ae} \Gamma_{ec}^b - f^{be} \Gamma_{ec}^a, \quad (34)$$

where  $\Gamma_{ec}^b$  is the connection on the brane's world manifold, and using the covariant derivative defined according to<sup>1</sup>

$$D_a D_b \psi^\mu = \partial_a D_b \psi^\mu - \Gamma_{ab}^c D_c \psi^\mu + \Gamma_{\rho\sigma}^\mu D_b \psi^\sigma \partial_a X^\rho. \quad (35)$$

So we arrive at the following deviation vector equation:

$$\begin{aligned} & D_c (D^c \psi_\alpha + \partial^c X_\alpha \partial^b X^\mu D_b \psi_\mu - \partial^b X_\alpha \partial^c X^\mu D_b \psi_\mu \\ & - \partial^b X_\alpha \partial_b X^\mu D^c \psi_\mu) + R_{\alpha\sigma\rho}^\nu g_{\mu\nu} \partial^c X^\mu \partial_c X^\rho \psi^\sigma = 0, \end{aligned} \quad (36)$$

which can be written compactly as

$$D_c (f^{cb}{}_{\alpha\mu} D_b \psi^\mu) + R_{\alpha\sigma\rho}^\nu g_{\mu\nu} \partial^c X^\mu \partial_c X^\rho \psi^\sigma = 0, \quad (37)$$

where

$$f^{cb}{}_{\alpha\mu} \equiv f^{cb} g_{\alpha\mu} + (f^{cd} f^{ab} - f^{ac} f^{bd} - f^{ad} f^{bc}) \partial_d X_\alpha \partial_a X_\mu. \quad (38)$$

Since the covariant derivative of the metric vanishes,  $D_c f^{ab} = 0$ , Eq. (36) or (37) can be written in the form

$$\begin{aligned} & (g_{\alpha\mu} - \partial^a X_\alpha \partial_a X_\mu) D_c D^c \psi^\mu - 2 D_c D^a X_\alpha \partial^c X_\mu D_a \psi^\mu \\ & + R_{\alpha\sigma\rho}^\nu g_{\mu\nu} \partial^c X^\mu \partial_c X^\rho \psi^\sigma = 0. \end{aligned} \quad (39)$$

<sup>1</sup> We distinguish the two kinds of connection by indices only. In this case, such simplified notation does not lead to confusion.

### Special case: point particle

Let us now consider a special case, in which the dimensionality of the brane's world manifold is  $n = 1$ , i.e., the case of a point particle. Then Eq. (36) becomes:

$$f^{00} D_0(D_0\psi_\alpha - \partial_0 X_a \partial_0 X^\mu f^{00} D_0\psi_\mu) + R^\nu{}_{\alpha\sigma\rho} g_{\mu\nu} f^{00} \partial_0 X^\mu \partial_0 X^\rho \psi^\sigma = 0. \quad (40)$$

Now  $f^{00} = \frac{1}{\dot{\tau}^2}$ ,  $f_{00} = \partial_0 X^\mu \partial_0 X^\nu g_{\mu\nu} \equiv \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$ . So we have

$$\frac{D}{D\tau} \left[ (g_{\alpha\beta} - \frac{\dot{X}_\alpha \dot{X}_\beta}{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}) \frac{D}{D\tau} \psi^\beta \right] + R^\nu{}_{\alpha\sigma\rho} g_{\mu\nu} \dot{X}^\mu \dot{X}^\rho \psi^\sigma = 0. \quad (41)$$

This is the point particle geodesic deviation equation derived by Bazanski [8, 9].

Our action is invariant under reparametrizations of the world manifold coordinates  $\xi^a$ . So there is a freedom to choose a gauge. In the case of the point particle, we can choose a gauge such that

$$\frac{\dot{X}^\mu}{\sqrt{\dot{X}^2}} \frac{D\psi_\mu}{D\tau} = \text{constant}. \quad (42)$$

Then Eq. (42) simplifies to

$$\frac{D^2\psi_\alpha}{D\tau^2} + R^\nu{}_{\alpha\sigma\rho} g_{\mu\nu} \dot{X}^\mu \dot{X}^\rho \psi^\sigma = 0. \quad (43)$$

Such equation we had in section 2 where we used the gauge in which  $\tau$  was equal to the proper time  $s$ .

### 3.1. Charged and spinning branes

In order to describe a *charged brane* moving in a gravitational and electromagnetic background field, let us generalize the Lagrangian (24) as follows:

$$L = \kappa\sqrt{f} f^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \Psi + e^\alpha F_{\mu\nu} \Psi^\mu \partial_\alpha X^\nu, \quad (44)$$

where  $e^\alpha$  is the charge current density on the brane [22]. Applying the variation with respect to the deviation vector, we obtain

$$D_c(f^{ac} \partial_c X^\mu) = \frac{e^\alpha}{\kappa\sqrt{f}} F^\mu{}_\nu \partial_\alpha X^\nu. \quad (45)$$

In the case of *spinning (rotating) brane*, the corresponding path equation which is the counterpart of Papapetrou

equation of spinning particles, is obtained from the Lagrangian

$$L = \kappa\sqrt{f} f^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \Psi + \frac{1}{2} f^{ab} R_{\mu\nu\rho\sigma} \Psi^\nu \partial_a X^\mu S_b^{\rho\sigma}. \quad (46)$$

Here  $S_a^{\rho\sigma}$  is the spin density current on the brane, such that the integration over the brane's world sheet surface element  $d\Sigma^a$  gives the total spin,  $S^{\rho\sigma}$ , of the brane:

$$\int d\Sigma^a S_a^{\rho\sigma} = S^{\rho\sigma}. \quad (47)$$

An example of such rotating or spinning brane is considered in ref. [19], where the extrinsic curvature term is added to the minimal surface action. The brane's spin then arises from the extrinsic curvature.

The variation of (46) with respect to the deviation vector on the brane gives

$$D_a(f^{ab} \partial_b X^\mu) = \frac{1}{2\kappa\sqrt{f}} f^{ab} R^\mu{}_{\nu\rho\sigma} \partial_a X^\nu S_b^{\rho\sigma}. \quad (48)$$

The above equation of motion is a particular case of the equation that was derived in ref. [19] for the brane with extrinsic curvature.

The deviation equation for *the charged brane* can be obtained by taking the variation of (44) with respect to  $X^\mu$ . So we obtain:

$$D_c(f^{cb\alpha\mu} D_b \psi_\mu) = R^\alpha{}_{\mu\nu\rho} \partial_a X^\mu \partial^a X^\nu \Psi^\rho + \frac{e^\alpha}{\kappa\sqrt{f}} (F^\alpha{}_\nu D_a \Psi^\nu + F^\alpha{}_{\nu;\rho} \partial_a X^\nu \Psi^\rho) \quad (49)$$

Similarly, by taking the variation of (46) with respect to  $X^\mu$ , we obtain the deviation equation for spinning brane:

$$D_c(f^{cb\alpha\mu} D_b \psi_\mu) = R^\alpha{}_{\mu\nu\rho} \partial_a X^\mu \partial^a X^\nu \Psi^\rho + \frac{1}{2\kappa\sqrt{f}} (R^\alpha{}_{\mu\nu\rho} S_a^{\nu\rho} D^a \Psi^\nu + R^\alpha{}_{\mu\nu\lambda} S_a^{\mu\lambda}{}_{;\rho} \partial^a X^\nu \Psi^\rho + R^\alpha{}_{\mu\nu\lambda;\rho} S_a^{\nu\lambda} \partial^a X^\mu \Psi^\rho). \quad (50)$$

## 4. Kaluza-Klein approach

Instead of considering a charged object in 4-dimensions, we can consider a neutral object in 5-dimensional curved spacetime with metric  $G_{MN}$ ,  $M, N = 0, 1, 2, 3, 5$ . In the case of a *point particle*, the Bazanski Lagrangian reads

$$L = M G_{MN} U^M \frac{D\psi^N}{DS}. \quad (51)$$

Here, constant  $M$  is the mass in 5-dimensions,  $U^M \equiv dX^M/dS$ ,  $\psi^N$  the geodesic deviation vector, and  $D\psi^N/DS = d\psi^N/dS + \hat{\Gamma}_{JK}^N \psi^J U^K$  is the covariant derivative with respect to the 5-dimensional line element  $dS = (G_{MN}dX^M dX^N)^{1/2}$ .

For the 5-dimensional metric tensor we will take the following Ansatz:

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\nu \\ A_\mu & 1 \end{pmatrix} \quad (52)$$

This is a simplified version of a more general 5D metric that, besides the 4D metric  $g_{\mu\nu}$  and the vector field  $A_\mu$ , also contains the scalar field  $\phi = G_{55}$ .

Using the relations

$$\psi_5 = G_{5N}\psi^N = \psi^5 + A_\nu\psi^\nu, \quad (53)$$

$$U_5 = G_{5N}U^N = U^5 + A_\nu\psi^\nu \quad (54)$$

the Lagrangian (51) can be written in the form

$$L = M \left( g_{\mu\nu} U^\mu \frac{D\psi^\nu}{DS} + U_5 A_\nu \frac{D\psi^\nu}{DS} + U_5 \frac{D\psi^5}{DS} \right). \quad (55)$$

Further, if we take into account<sup>2</sup>  $DA_\nu/DS = 0$ , then we have

$$L = M \left( g_{\mu\nu} U^\mu \frac{D\psi^\nu}{DS} + U_5 \frac{D\psi_5}{DS} \right). \quad (56)$$

The components of the 5-dimensional connection split according to

$$\hat{\Gamma}_{\nu\lambda}^\mu = \Gamma_{\nu\lambda}^\mu + \frac{1}{2}(A_\lambda F_{\nu}{}^\mu + A_\nu F_{\lambda}{}^\mu), \quad (57)$$

$$\hat{\Gamma}_{\mu\nu}^5 = \frac{1}{2}(\nabla_\mu A_\nu + \nabla_\nu A_\mu) - \frac{1}{2}A^\rho(A_\nu F_{\mu\rho} + A_\mu F_{\nu\rho}), \quad (58)$$

$$\hat{\Gamma}_{5\nu}^\mu = \frac{1}{2}F_{\nu}{}^\mu, \quad \hat{\Gamma}_{5\mu}^5 = -\frac{1}{2}A^\nu F_{\mu\nu}, \quad \hat{\Gamma}_{55}^\mu = 0, \quad \hat{\Gamma}_{55}^5 = 0. \quad (59)$$

From

$$\frac{D\psi^\nu}{DS} = \frac{d\psi^\nu}{dS} + \hat{\Gamma}_{JK}^\nu \psi^J U^K, \quad (60)$$

$$\frac{D\psi_5}{DS} = \frac{d\psi_5}{dS} - \hat{\Gamma}_{5N}^M \psi_M U^N, \quad (61)$$

and  $\psi_M = G_{MJ}\psi^J$ , i.e.,  $\psi_\mu = G_{\mu\rho}\psi^\rho + G_{\mu 5}\psi^5$ ,  $\psi_5 = G_{55}\psi^5 + G_{5\mu}\psi^\mu$ , where  $G_{\mu\rho} = g_{\mu\rho} + A_\mu A_\rho$ , we find

$$\frac{D\psi^\nu}{DS} = \frac{d\psi^\nu}{dS} + \Gamma_{\rho\sigma}^\nu \psi^\rho U^\sigma + \frac{1}{2}U_5 F_{\rho}{}^\nu \psi^\rho \quad (62)$$

$$\frac{D\psi_5}{DS} = \frac{d\psi_5}{dS} - \frac{1}{2}F_{\nu}{}^\mu U^\nu g_{\mu\rho} \psi^\rho \quad (63)$$

The Lagrangian (56) thus becomes

$$L = M g_{\mu\nu} U^\mu \left( \frac{d\psi^\nu}{dS} + \Gamma_{\rho\sigma}^\nu \psi^\rho U^\sigma \right) + M U_5 F_{\rho\mu} \psi^\rho U^\mu + M U_5 \frac{d\psi_5}{dS}, \quad (64)$$

and the action is

$$I = \int L dS = \int L \frac{dS}{ds} ds = \int ds \left[ m g_{\mu\nu} u^\mu \left( \frac{d\psi^\nu}{ds} + \Gamma_{\rho\sigma}^\nu \psi^\rho u^\sigma \right) + e F_{\rho\mu} u^\mu + e \frac{d\psi_5}{ds} \right]. \quad (65)$$

where

$$e = M U_5 = M \frac{dX_5}{dS} = m \frac{dX_5}{ds}, \quad ds = (g_{\mu\nu} dX^\mu dX^\nu)^{1/2}, \quad (66)$$

$$m = M \frac{ds}{dS} \quad \text{and} \quad u^\mu = \frac{dX^\mu}{ds}. \quad (67)$$

The Lagrangian in Eq. (65) is just like that in Eq. (6), apart from the additional term  $e d\psi_5/ds$ . One must also bear in mind that  $e = M U_5 = M(U^5 + A_\nu U^\nu)$ , and that because  $e$  depends on  $U^\nu = dX^\nu/dS$ , it contributes to the variation of the action with respect to  $X^\nu$ , so that besides the terms occurring in Eq.(13) we obtain the additional term

$$F_{\lambda}{}^\alpha U^\lambda \left[ \frac{d\psi_5}{dS} + F_{\rho\mu} \psi^\rho U^\mu \right]. \quad (68)$$

If we perform the variation with respect to  $X^5$ , we obtain the deviation equation for the fifth component of the deviation vector:

$$\frac{d}{dS} \left[ \frac{d\psi_5}{dS} + F_{\rho\mu} \psi^\rho U^\mu \right] = 0. \quad (69)$$

The latter equations are in agreement with the equations derived by Kerner et al.[20] directly from the geodesic deviation equation in five dimensions. Kerner et al.[20] observed that the expression in square brackets of Eq. (68)

<sup>2</sup> Recall that  $A_\nu = G_{5\nu}$ , i.e., some components of the metric.

satisfies equation (69), and is thus a constant of motion. In particular, if the latter constant of motion is zero, then we have a one-to-one correspondence between the geodesic deviation equation in 5 dimensions, derived from the action (65) and the usual deviation equation (13) in presence of the electromagnetic field in 4 dimensions. Those results hold for a particular Ansatz (52) with  $G_{55} = 1$ . In the case of a  $p$ -brane, the Lagrangian (24) can be extended to

$$L = \kappa \sqrt{f} f^{ab} G_{MN} \partial_a X^M D_b \Psi^N, \quad (70)$$

where  $M, N = \mu, 5$ . A similar procedure as for the point particle gives

$$L = \kappa \sqrt{f} f^{ab} [g_{\mu\nu} \partial_a (\partial_b \Psi^\nu + \Gamma_{\rho\sigma}^\nu \Psi^\rho \partial_b X^\sigma) + \partial_5 X_5 F_{\rho\mu} \Psi^\rho \partial_b X^\mu + \partial_a X_5 \partial_b \Psi_5], \quad (71)$$

which can be written as

$$L = \kappa \sqrt{f} f^{ab} g_{\mu\nu} \partial_a X^\mu (\partial_b \Psi^\nu + \Gamma_{\rho\sigma}^\nu \Psi^\rho \partial_b X^\sigma) + e^a F_{\rho\mu} \Psi^\rho \partial_a X^\mu + e^a \partial_a \Psi_5, \quad (72)$$

where  $e^a \equiv \kappa \sqrt{f} f^{ab} \partial_b X_5$ . We see that the Lagrangian (44) is embedded in the Lagrangian (44) which comes from five dimensions. Instead of five, we can consider more dimensions of the embedding spacetime.

## 5. Discussion and conclusion

Since branes are so important objects considered in the attempts to develop a unified theory of fundamental interactions, quantum gravity and the brane world cosmological models, we feel that such tasks require a thorough knowledge of all aspects of the brane theory, starting with the most basic ones such as the classical equations of motion. We have shown how the path equations of motion for branes and the corresponding deviation equations can be obtained from a single Lagrangian (24), which is a generalization of the Bazanski Lagrangian. If a physical system consists of many branes, then their relative motion can be described in terms of the deviation equations. We have considered the minimal surface deviation equation (37), and the deviation equations for charged and spinning (rotating) branes.

The Bazanski action in 5-dimensions can be split into the action for a charged object (point particle or a brane) in 4-dimensions plus the extra terms. The latter terms enable distinction between the 4-dimensional and the 5-dimensional or, in general, a higher dimensional, theory.

Deviation equations for charged point particles and branes can thus be used in testing the presence of extra dimensions.

A possible extension of the  $p$ -brane path and path deviation equations is an analogous set of equations in the brane configuration space [21], a theoretical framework which has a potential to explain a deeper geometric principle behind brane theory.

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