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Solutions of Fractional Order Model of Childhood diseases with Constant Vaccination Strategy

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Abstract: Childhood vaccination programs have had a dramatic impact on child morbidity and mortality. Protecting children from diseases that can be prevented by vaccination is a primary goal of health administrators. A SIR model that monitors the temporal dynamics of a childhood disease in the presence of preventive vaccine is presented in this paper. We introduce fractional-order into the presented model. Homotopy analysis method (HAM) is considered in this paper to obtain an analytic approximate solution of this model. The results obtained by HAM are compared with the classical fourth order Runge–Kutta method (RK4) to gauge its effectiveness. The obtained results proved that the disease will persist within the population if the vaccination coverage level is below a certain threshold.

Keywords: Infectious diseases modes, Fractional order differential equations, Homotopy analysis method.

1 Introduction

Childhood infectious diseases [1] like measles, mumps, rubella, poliomyelitis, chicken pox, are important public health problem. Since vaccination is considered to be the most effective strategy against childhood diseases, the development of a framework that would predict the optimal vaccine coverage level needed to prevent the spread of these diseases is crucial. A universal effort to extend vaccination coverage to all children began in 1974, When the World Health Organization (WHO) founded the Expanded Program on Immunization (EPI). Mathematical models, of deterministic type, have often been used to provide deeper insights into the transmission dynamics of childhood diseases and to evaluate control strategies. In the SIR model presented in this paper, the population that is involved in the spread of an infection is split into three epidemiological classes: a susceptible group (S), an infected group (I), and a removed group (R) denoting vaccinated as well as recovered people with permanent immunity. This model assumes that the efficacy of the vaccine is 100% and the natural death rates μ in the classes remain unequal to births, so that the population size N is realistically not constant. Citizens are born into the population at a constant birth rate π with extremely very low childhood disease mortality rate. We denote the fraction of citizens vaccinated at birth each year as P (with $0 < P < 1$) and assume the rest are susceptible. A susceptible individual will move into the infected group through contact with an infected individual, approximated by an average contact rate β. An infected individual recovers at a rate γ, and enters removed group. The removed group [2] also contains people who are vaccinated. The differential equations for the SIR model are

$$\begin{aligned} \frac{dS}{dt} &= (1-P)\pi N - \beta\frac{SI}{N} - \mu S, \\ \frac{dI}{dt} &= \beta\frac{SI}{N} - (\gamma+\mu)I, \\ \frac{dR}{dt} &= P\pi N + \gamma I - \mu R \end{aligned} \tag{1}$$

Where N=S+I+R, and the parameters μ, π, β, γ are assumed to be positive. By adding the three equations of the above system (1), we obtain

$$\frac{dN}{dt} = (\pi - \mu)N,$$

so that we are now dealing with a varying total population. A summary of the process is presented through (2). The groups can be scaled by population N using the new variables:

$s = S/N, i = I/N, r = R/N$. The population is now normalised, meaning $s + i + r = 1$, and we have the new system:

$$\frac{ds}{dt} = (1-P)\pi - \beta si - \pi s,$$
$$\frac{di}{dt} = \beta si - (\gamma + \pi)i, \quad (2)$$
$$\frac{dr}{dt} = P\pi + \gamma i - \pi r$$

The rest of the paper is organized as follows. In section 2, a discussion about the fractional calculus is of the presented, while section 3 gives an idea about the Homotopy analysis method for solving fractional order ordinary differential equations. The fractional order is introduced to the system (2) in section 4. Section 5 is devoted to the numerical results of the presented problem.

1. Fractional calculus

Fractional calculus (FC) has been extensively applied in many fields [3-5]. Many mathematicians and applied researchers have tried to model real processes using the fractional calculus One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives. Another difficult is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character. It was found that various; especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. For the concept of fractional derivative [6], we will adopt Caputo's definition, which is a modification of the Riemann–Liouville definition and has the advantage of dealing properly with initial value problems.

Definition 1 The fractional integral of order $\alpha > 0$ of a function $f: R^+ \to R$ is given by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \quad (3)$$

Where $\alpha > 0,\ x > 0,\ J^0 f(x) = f(x)$.
Hence we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma} \quad (4)$$

Where $\alpha > 0, \gamma > -1, t > 0$

Definition 2 Riemann–Liouville and Caputo fractional derivatives of order α of a continuous function $f: R^+ \to R$ is given respectively by

$$D_*^\alpha f(x) = D^m\left(J^{m-\alpha} f(x)\right),$$

$$D^\alpha f(x) = J^{m-\alpha}\left(D^m f(x)\right),$$

where $m-1 < \alpha \leq m, m \in N$.

The definition of fractional derivative involves an integration which is non local operator (as it is defined on an interval) so fractional derivative is a non local operator. In other word, calculating time-fractional derivative of a function $f(t)$ at some time $t = t_1$requires all the previous history, i.e. all $f(t)$ from $t = 0\ to\ t = t_1$.

2. Homotopy analysis method (HAM)

Many of the (FDEs) that arise in physical or biological situations are highly non-linear. As a result, it is often difficult to obtain analytical solutions to these problems [7]. Some of the recent analytic methods for solving nonlinear problems include the Adomian decomposition method (ADM), homotopy-perturbation method (HPM), variational iteration method (VIM). Liao [8] gives an example that effectively illustrates the limitations of traditional perturbation methods: the problem of a body falling freely through . He then goes on to give an alternative technique known as the homotopy analysis method (HAM). The basic idea of the HAM method is to produce a succession of approximate solutions that tend to the exact solution of the problem. The presence of auxiliary parameters and functions in the approximate solution results in the production of a family of approximate HAM provides us with a simple way to adjust and control the convergence region of the series solution by introducing the auxiliary parameter $h \neq 0$, and the auxiliary function $H \neq 0$. One can get accurate

approximations by only a few terms with $h = -1$ and $H = 1$. Besides, the so-called ‘‘homotopy perturbation method’’ (proposed in 1998) is exactly the same as the early homotopy analysis method (proposed in 1992) and is a special case of the late homotopy analysis method in case of $h = -1$.

Consider the following [6] system of (FDE):

$$D^{\alpha_i}\big(u_i(t)\big) = f_i(t, u_1, \dots, u_n), \quad (5)$$

where $i = 1,2,3, \dots, n,\ \ 0 \le \alpha_i \le 1$

Subject to the initial conditions:

$$u_i(0) = a_i, \quad i = 1,2, \dots, n \quad (6)$$

Liao [9] constructed the so-called zeroth-order deformation equation:

$$(1-q)\mathcal{L}_i[\phi_i(t,q) - u_{i0}(t)] = q h_i H_i(t) N_i[\phi_i(t,q)] \quad (7)$$

where $i = 1,2,3, \dots .n,$

subject to the initial conditions:

$$\phi_i(0, q) = a_i \quad (8)$$

Where $q \in [0,1]$ is an embedding parameter, N_i are nonlinear operators, $\mathcal{L}_i$ are auxiliary linear operators satisfy$\mathcal{L}_i(0) = 0$, $u_{i0}(t)$ are initial guesses satisfy the initial conditions (6), $h_i \neq 0$ are auxiliary parameters, $H_i(t) \neq 0$ are auxiliary functions, $\phi_i(t,q)$are unknown functions. It should be emphasized that one has great freedom to choose, the auxiliary linear operators $\mathcal{L}_i$, the auxiliary parameters h_i and the auxiliary functions H_i. Obviously, when $q \neq 0$, since $u_{i0}(t)$ satisfy the initial conditions (6) and $\mathcal{L}_i(0) = 0$, we have

$$\phi_i(t, 0) = u_{i0}(t), \quad i = 1,2,3, \dots, n, \quad (9)$$

when $q = 1$, since $h_i \neq 0$ and $H_i(t) \neq 0$, the zeroth-order deformation equation (7) and (8) are equivalent to (5) and (6), hence

$$\phi_i(t, 1) = u_{i0}(t), \quad i = 1,2,3, \dots, n. \quad (10)$$

Thus, as q increasing from zero to one, the solutions $\phi_i(t,q)$ various from $u_{i0}(t)$to $u_i(t)$. Expanding $\phi_i(t,q)$ in Taylor series with respect to the embedding parameter q, one has

$$\phi_i(t,q) = u_{i0}(t) + \sum_{m=1}^{\infty} u_{im}(t) q^m \quad (11)$$

where $i = 1,2,3, \dots, n,$

$$u_{im}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t,q)}{\partial q^m}\Big|_{q=0} \quad (12)$$

where $i = 1,2,3, \dots, n$

Assume that the auxiliary parametersh$_i$, the auxiliary functions $H_i(t)$, the initial approximations $u_{i0}(t)$ and the auxiliary linear operators $\mathcal{L}_i$ are properly chosen so that the series (11) converges at $q = 1$. Then at $q = 1$, and by (10) the series (11) becomes

$$u_i(t) = u_{i0}(t) + \sum_{m=1}^{\infty} u_{im}(t)\ , \quad (13)$$

$i = 1,2,3, \dots, n.$

and now define the vector

$$\vec{u}_i = \{u_{i0}, u_{i1,} u_{i2}, \dots, u_{ij}\}, \quad (14)$$

$i = 1,2,3, \dots j.$

Differentiating equations (7) m times with respect to the embedding parameter q, then setting $q = 0$ and dividing them by m!, finally using (12), we have the so-called mth-order deformation equations:

$$\mathcal{L}_i\big[u_{im} - \mathcal{X}_m u_{i(m-1)}(t)\big] = h_i H_i(t) \Re_{im}\left(\vec{u}_{i(m-1)}(t)\right), \quad (15)$$

Where $i = 1,2, \dots, n,$

Subject to the conditions:

$$u_{im}(0) = 0, \quad (16)$$

$i = 1,2, \dots, n, m = 1,2,3, \dots n,$

Where

$\Re_{im}\left(\vec{u}_{i(m-1)}(t)\right)$

$$= \frac{1}{(m-1)!}\frac{\partial^{m-1}N_i\left(\phi_i(t,q)\right)}{\partial q^{m-1}}\Big|_{q=0} \tag{17}$$

and

$$\mathcal{X}_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{18}$$

If we choose the linear operator $\mathcal{L}_i = \mathrm{D}^{\alpha_i}$ then according to (15), we have

$$J^{\alpha_i}D^{\alpha_i}\left[u_{im} - \mathcal{X}_m u_{i(m-1)}(t)\right] = h_i J^{\alpha_i}\left[H_i(t)\Re_{im}\left(\vec{u}_{i(m-1)}(t)\right)\right] \tag{19}$$

Finally it seems that, as long as a nonlinear fractional order differential equation has at least one solution, then one can always construct a kind of zeroth-order deformation equation to get convergent homotopy-series solution as

$$u_{im} = \mathcal{X}_m u_{i(m-1)}(t) + h_i J^{\alpha_i}\left[H_i(t)\Re_{im}\left(\vec{u}_{i(m-1)}(t)\right)\right] \tag{20}$$

and $u_i = u_{i0} + u_{i1} + u_{i2} + u_{i3} + u_{i4} + \cdots$

1. Fractional model derivation

Now we introduce fractional-order into the model (1). The new system is described by the following set of FDEs of order $\alpha_1, \alpha_2, \alpha_3 > 0$:

$$\begin{aligned} D^{\alpha_1}(s) &= (1-P)\pi - \beta si - \pi s, \\ D^{\alpha_2}(i) &= \beta si - (\gamma + \pi)i, \\ D^{\alpha_3}(r) &= P\pi + \gamma i - \pi r \end{aligned} \tag{21}$$

This paper attempts to find a numerical solution for a general class of fractional order model of childhood diseases (21). For this purpose the paper summarizes specific techniques for homotopy analysis method (HAM), as well as the applications of Caputo fractional calculus. The reason of using fractional order differential equations (FOD) is that FOD are naturally related to systems with memory which exists in most biological systems. They are closely related to fractals which are abundant in biological systems. Also the Fractional order differential equations are, at least, as stable as their integer order counterpart. The results derived of the fractional system (21) are of a more general nature. We would like to put your attention that time fractional derivatives change also the solutions we usually get in standard system (2). Consequently, from (20) and if we select the auxiliary functions $H_1 = H_2 = H_3 = 1$,we have the HAM series solution of the given system (21) can be given by:

$s = s_0 + s_1 + s_2 + s_3 + s_4 + \cdots$
$i = i_0 + i_1 + i_2 + i_3 + i_4 + \cdots$
$r = r_0 + r_1 + r_2 + r_3 + r_4 + \cdots$

HAM yields rapidly convergent series solutions by using a few iterations for both linear and non-linear deterministic equations. Nevertheless, HAM has a great freedom to adjust and control the convergence region of solution series by an auxiliary parameter h. We still have freedom to choose the auxiliary parameters h_1, h_2, h_3 from the so called h-curves. For$\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $h_1 = h_2 = h_3 = -1$ we obtained four terms approximations which is the same solutions obtained in [1,2] using homotopy perturbation method (HPM) and Adomian decomposition method (ADM).

Case	s_0	i_0	r_0	β	γ	π	P	R_v
1st	1	0	0	0.8	0.03	0.4	0.9	0.18
2nd	0.8	0.2	0	0.8	0.03	0.4	0	1.86

Table:1

1. Numerical results and discussion

In this section, we will study the effect of vaccination on the dynamics of a childhood disease described by the SIR model (21) using HAM according to the different values of the parameters in table 1. Firstly, assume $\alpha_1 = \alpha_2 = \alpha_3 = 1$. The values of h can be taken from the so called h-curves.

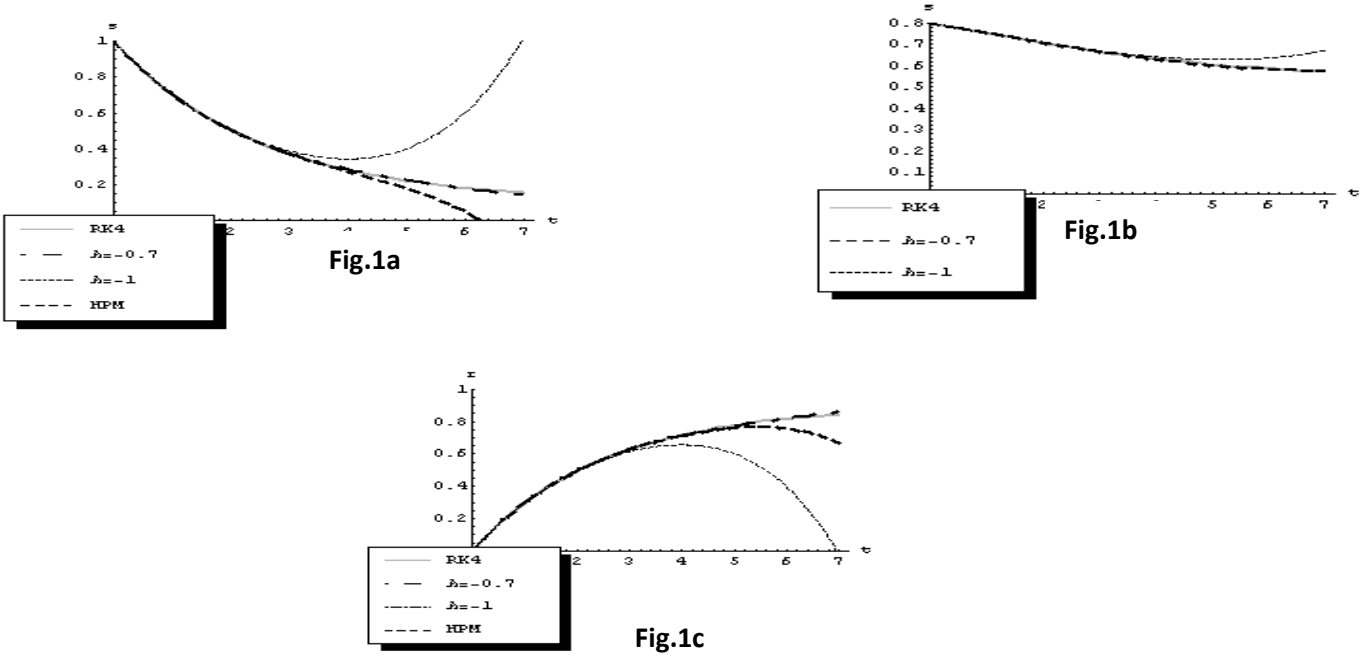

Fig.1: The population fraction in the 1st Case.

Of course the accuracy and the convergence can be improved by computing more terms. In this paper by using HAM we can control the convergence by controlling the value of h. Fig.1 presents the solution of the presented model in the first case when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 1$ for $h_1 = h_2 = h_3 = h$. If $h = -1$, the obtained 4th order solution series by HAM is the same solution by HPM [1] and ADM [2]. From the so called h-curves, if $h = -0.7$, the results of the 4th order solutions obtained by HAM are in excellent agreement with those of the RK4 method and better than the 6th order solutions obtained by HPM and ADM. In other words by controlling the value of h in HAM, we can reduce the long calculations and obtain an accurate solution by only few terms. Fig 2 presents the second case of the presented problem for different values of h. Fig 2 also presents the solution of (21) at different values of α and $h = -0.7$. Fig.2 presents the solution of the presented model in the first case when $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 1$for $h_1 = h_2 = h_3 = h$.

Fig.2: The population fraction in the 2nd case

If $h = -1$, the obtained 4th order solution series by HAM is the same solution by HPM [1] and ADM [2]. From the so called h-curves, if $h = -0.7$, the results of the 4th order solutions obtained by HAM are in excellent agreement with those of the RK4 method and better than the 6th order solutions obtained by HPM and ADM.

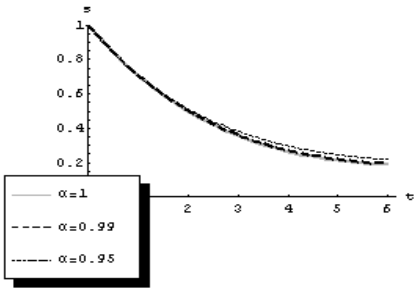

Fig.3a

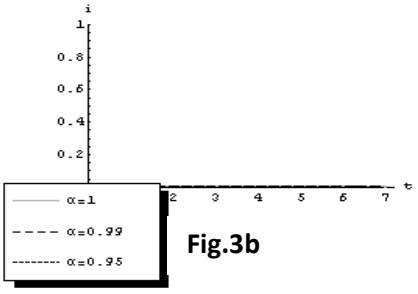

Fig.3b

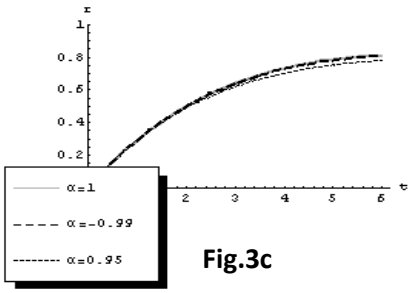

Fig.3c

Fig.3: The population fraction vs. time in the 1st case for different valued of α.

In other words by controlling the value of h in HAM, we can reduce the long calculations and obtain an accurate solution by only few terms. Fig 2 presents the second case of the presented problem for different values of h. Fig 3 presents the solution of (21) at different values of α and $h = -0.7$.

3. Conclusion

In this paper, HAM was implemented to describe the effect of vaccination on the dynamics of a childhood disease described by the fractional SIR model (21). The results show that the solution continuously depends on the time-fractional derivative and on the values of the parameters described in table 1. Its results are in excellent agreement with RK4. Fig.2 describes Case 1 (the first three curves) and shows the effect of high-vaccination coverage ($P > P_c$) on the disease free initial population groups. The population of the susceptible group decreases with time while that of the removed group gradually increases due to inclusion of vaccinated susceptible group. The entire population generally remains disease free with all the time and the endemic equilibrium remains stable. Case 2 is shown in Fig.2 and illustrates the impact of initial low levels of infective group on the vaccination free population ($P < P_c$). As expected, the population of susceptible group decreases while that of infective group temporally increases. The disease rapidly spread to the entire population. The only contribution to removed group is the very small proportion of recovered people with permanent immunity. It is observed that the disease free equilibrium is stable provided the vaccination coverage level exceeds a certain threshold $P_c = (\beta - \gamma - \pi)/\beta = 0$. As a definition of fractional calculus: $lim_{\alpha \to 1} D^{\alpha} f(t) = Df(t)$ has been provided. In the presented problem, the susceptible group s(t), the infected group i(t), and the removed group r(t), have been obtained, the results obtained show that when $\alpha \to 1$ (Fig.3) the solution of the fractional model (18) $s_{\alpha}, i_{\alpha}, r_{\alpha}$, reduce to the standard solution $s(t)$, $i(t), r(t)$.

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