Mathematical Problems in Engineering,
Volume 7, pp. 455-484
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Passivity Analysis and Synthesis for Uncertain Time-Delay Systems

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(Received 3 October 2000; In final form March 2001)

In this paper, we investigate the robust passivity analysis and synthesis problems for a class of uncertain time-delay systems. This class of systems arises in the modelling effort of studying water quality constituents in fresh stream. For the analysis problem, we derive a sufficient condition for which the uncertain time-delay system is robustly stable and strictly passive for all admissible uncertainties. The condition is given in terms of a linear matrix inequality. Both the delay-independent and delay-dependent cases are considered. For the synthesis problem, we propose an observer-based design method which guarantees that the closed-loop uncertain time-delay system is stable and strictly passive for all admissible uncertainties. Several examples are worked out to illustrate the developed theory.

Keywords: Design; Observer; Passivity; Uncertainties; Time-delay systems; Water quality

1 INTRODUCTION

Stability analysis and control problems for dynamical systems with delay factors in the state variables and/or control inputs have received considerable interests for more than three decades [1,2]. Recently, output-feedback control schemes have been developed for the stabilization of a wide-class of time-delay systems; see [3–5,10] and the references cited therein. Different issues and approaches related to uncertain time-delay systems are thoroughly discussed in

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[9,11] along with vast breadth of techniques. On another front of research, positive real theory has played a major role in stability and systems theory [6,7,17]. The primary motivation for designing strict positive real controllers is for applications to positive real plants. When a strict positive real system is connected to a positive real plant in a negative-feedback configuration, the closed-loop is guaranteed to be stable for arbitrary plant variations as long as the plant remains to be positive real. Recently, state-space formulae for the controller synthesis have been developed in [8] to guarantee that the closed-loop transfer function is exteded strictly positive real. Some related work for a class of uncertain dynamical systems are reported in [18–20]. In [16], conditions for positive realness for a class of linear time-invariant systems with state-delay have been developed.

In this work, we contribute to the further developement of passive analysis and control synthesis of a class of uncertain time-delay systems. This class of systems arises in the modelling effort of studying water quality constituents in fresh stream for which all the analytical results and numerical simulation in the sequel are focused. We restrict attention on linear time-invariant models and this, in turn, facilitates the use of the fact that the notion of positive realness is closely related to the passivity of linear time-invariant systems thereby extending the results of [6–8]. One main result in this work is the derivation of a sufficient condition for which the uncertain time-delay system under consideration is robustly stable and strictly passive for all admissible uncertainties. The condition is given in terms of a linear matrix inequality the solution of which can be readily obtained by an efficient software package [12]. In our work, the analysis of both the delay-independent and delay-dependent cases are considered. For the synthesis problem, we propose an observer-based design method which guarantees that the closed-loop uncertain time-delay system is stable and strictly passive for all admissible uncertainties. Several numerical examples of water quality models are worked out to illustrate the theoritcal results.

Notations and Facts. In the sequel, we denote by W^t, W^{-1} and $\lambda(W)$ the transpose, the inverse and the eigenvalues of any square matrix W. $W > 0$ ($W < 0$) stands for a positive- (negative-) definite matrix W and $\mathcal{C}^-$ represents the open left-half of the complex plane. Sometimes, the arguments of a function will be omitted when no confusion can arise.

Fact 1 (Schur Complement) Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$, then $\Omega_1 + \Omega_3^t \Omega_2^{-1} \Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix} < 0$$

Fact 2 For any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions, such that $0 < \Sigma_3 = \Sigma_3^t$, it follows that

$$\Sigma_1^t \Sigma_2 + \Sigma_2^t \Sigma_1 \leq \Sigma_1^t \Sigma_3 \Sigma_1 + \Sigma_2^t \Sigma_3^{-1} \Sigma_2$$

In the case when Σ_3 reduces to a scalar $\alpha > 0$, the above inequality reduces to:

$$\Sigma_1^t \Sigma_2 + \Sigma_2^t \Sigma_1 \leq \alpha \Sigma_1^t \Sigma_1 + \alpha^{-1} \Sigma_2^t \Sigma_2, \quad \alpha > 0$$

2 MOTIVATING SYSTEM MODEL

In every aspect of life, it is important to keep water quality in streams at a standard level. This can be measured by the concentrations per unit volume of some water biochemical constituents like algae, nitrogen components, phosphate, biological oxygen demand and dissolved oxygen. In building up dynamic models of water quality systems, it is customary to introduce some simplifying assumptions. Thus we consider that the stream has a constant flow rate, the flow of water is turbulence-free and the water is well mixed. We further assume that there exists $\tau > 0$ such that the concentrations of the water quality constituents entering at time t are equal to the corresponding concentrations τ time units ago. The basic tools for mathematical modeling are mass balance and some appropriate physio-chemical and biological expressions for the growth of water quality constituents. Recent water-quality studies on the River Nile for the purpose of dynamic modeling and control have shown [13–15] that a reduced second-order aggregate model for a small representative reach would be sufficient to give a satisfactory performance. In a typical model, the state-variables are the concentrations of pollutants P_A (representing a mixture of the low-levels in the bio-strata) and pollutant P_B (representing a mixture of the other levels in the bio-strata). The control variables are signals proportional to the water speed and the amount of effluent discharged into the reach at pre-selected points. The growth model takes the form:

$$\dot{x}(t) = f[x(t), u(t), x(t-\tau(t)), p(t)] \tag{1}$$

where $p(t)$ is a vector representing various rate coefficients and $x(t-\tau(t))$ stands for the level of water quality constituents τ time units ago. By linearizing (1) about a desired steady state point (X^*, U^*) we obtain the dynamic model

$$\delta\dot{X}(t) = A\delta X(t) + G_o\delta U(t) + E_o\delta X(t-\tau(t)) + Z(t) \tag{2}$$

where,

$$A = \frac{\partial f}{\partial x(t)}\Big|_*, \quad G_o = \frac{\partial f}{\partial u(t)}\Big|_*, \quad E_o = \frac{\partial f}{\partial x(t-\tau(t))}\Big|_*.$$

The elements of A are the average rate coefficients, those of B_o are the input signal coefficients and $\tau(t)$ reflects the extent within the reach that physico-chemical reactions affect the concentration at a prescribed position. The matrix E_o represents coefficients of physico-chemical interactions over time. The $Z(t)$ term lumps together the effect of high-order terms and it is usually not completely known but bounded. In the present analysis, we drop out the $Z(t)$ term, let $x = \delta X$ and $u = \delta U$ and consider the model variability as uncertainties in system parameters. That is, we let $A = A_o + \Delta A$ and in this case the water quality model will be of the form:

$$\begin{aligned}\dot{x}(t) &= (A_o + \Delta A)x(t) + G_o u(t) + E_o x(t-\tau(t)) \\ &= A_\Delta x(t) + G_o u(t) + E_o x(t-\tau) \end{aligned} \tag{3}$$

where ΔA stands for the parameteric uncertainties. Model (3) represents a class of linear uncertain systems with state-delay and it is purpose of this paper to provide a detailed analysis and synthesis for this class of systems based on passivity theory.

3 A CLASS OF UNCERTAIN SYSTEMS

To proceed with the stability analysis, we set $u(t) \equiv 0$ in (3) and add up a performance input-output pair to yield:

$$\begin{aligned}(\Sigma_\Delta) : \dot{x}(t) &= A_\Delta x(t) + B_o w(t) + E_o x(t-\tau) \\ &= (A_o + \Delta A)x(t) + B_o w(t) + E_o x(t-\tau)\end{aligned} \tag{4}$$

$$z(t) = C_\Delta x(t) + D_o w(t) = (C_o + \Delta C) + D_o w(t) \tag{5}$$

where the performance pair is given by $w \in \Re^p$ as the exogenous input and $z \in \Re^p$ as the output, $x \in \Re^n$ is the state , τ is a time delay, $B_o \in \Re^{n\times p}$ is a constant input matrix, $E_o \in \Re^{n\times n}$ is a constant delay matrix and $A_\Delta \in \Re^{n\times n}$, and $C_\Delta \in \Re^{p\times n}$ are uncetain matrices given by:

$$\begin{bmatrix} A_\Delta \\ C_\Delta \end{bmatrix} = \begin{bmatrix} A_o \\ C_o \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \Delta(t) E \tag{6}$$

with

$$\Delta^t(t)\Delta(t) \leq I \quad \forall\, t \tag{7}$$

Note that model (4)–(5) allows for unpredictable changes in the water quality parameters (term ΔA) as well as unknown measurement errors (term ΔC). Distinct from system (4)–(5) is two systems: 1) The free nominal system:

$$(\Sigma_f)\text{: } \dot{x}(t) = A_o x(t) + E_o x(t-\tau) \tag{8}$$

$$z(t) = C_o x(t) \tag{9}$$

in which all uncertainties and the input w are suppressed and

2) The nominal system

$$(\Sigma_o)\colon \dot{x}(t) = A_o x(t) + B_o w(t) + E_o x(t-\tau) \tag{10}$$

$$z(t) = C_o x(t) D_o w(t) \tag{11}$$

Our purpose is to examine the problem of passive analysis of (Σ_Δ) in relation to (Σ_f) and (Σ_o).

4 CONDITIONS OF PASSIVITY

In this section, we provide some technical results on stability and passivity for a class of linear state-delay systems of the type (3) which will be used in the sequel.

4.1 Some Stability Conditions

ASSUMPTION 1 $\lambda(A_o) \in C^-$.

We have the following delay-independent stability result concerning the system (Σ_f).

LEMMA 1 *Subject to Assumption 1, the time delay system (Σ_f) is globally asymptotically stable independent-of-delay if one of the following two equivalent conditions holds:*

(1) *There exist matrices $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$ satisfying the linear matrix inequality (LMI)*

$$\begin{bmatrix} PA_o + A_o^t P + Q & PE_o \\ E_o^t P & -Q \end{bmatrix} < 0 \tag{12}$$

(2) *There exist matrices $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$ satisfying the algebraic Riccati inequality (ARI)*

$$PA_o + A_o^t P + PE_o Q^{-1} E_o^t P + Q < 0 \tag{13}$$

Proof See the Appendix.

In some applications where the upper bound for the unknown delay is known, delay-dependent stability results are desired. For those cases, another route of analysis has to be followed, which requires the following assumption:

ASSUMPTION 2 $\lambda(A_o + E_o) \in \mathcal{C}^-$.

Note that Assumption 2 corresponds to the stability condition when $\tau = 0$. Hence the assumption is necessary for the system (1) to be stable for any $\tau \geq 0$.

LEMMA 2 *Consider the time-delay system* (Σ_f) *satisfying Assumption 2. Then given a scalar* $\tau^* > 0$, *the system* (Σ_f) *is globally asymptotically stable for any constant time-delay* τ *satisfying* $0 \leq \tau \leq \tau^*$ *if one of the following two equivalent conditions holds:*

(1) *Given scalars* $\varepsilon > 0$ *and* $\alpha > 0$, *there exists a matrix* $0 < X = X^t \in \Re^{n \times n}$ *and satisfying the LMI:*

$$\begin{bmatrix} (A_o + E_o)X + X(A_o + E_o)^t + \tau^*(\varepsilon + \alpha)E_oE_o^t & \tau^* XA_o^t & \tau^* XE_o^t \\ \tau^* A_oX & -(\tau^*\varepsilon)I & 0 \\ \tau^* E_oX & 0 & -(\tau^*\alpha)I \end{bmatrix} < 0 \tag{14}$$

(2) *Given scalars* $\varepsilon > 0$ *and* $\alpha > 0$, *there exists a matrix* $0 < X = X^t \in \Re^{n \times n}$ *and satisfying the ARI*

$$\begin{aligned} &(A_o + E_o)X + (A_o + E_o)^t X + \tau^*\varepsilon^{-1}XA_o^tA_oX \\ &+ \tau^*\alpha^{-1}XE_o^tE_oX + \tau^*(\varepsilon + \alpha)E_oE_o^t < 0 \end{aligned} \tag{15}$$

Proof See the Appendix.

4.2 Example 1

In order to illustrate Lemma 1, we consider a water quality model of the form (3) with

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad E_o = \begin{bmatrix} 0 & 0.3 \\ 0.3 & -0.2 \end{bmatrix}$$

It is easy that $\lambda(A_o) = \{-1, -2\}$ which satisfies Assumption 1. Using the LMI solver of MATLAB Control Toolbox [12], it is found that the feasible solution of the LMI (11) is given by:

$$P = \begin{bmatrix} 21.2515 & 18.0847 \\ 18.0847 & 78.5135 \end{bmatrix}, \quad Q = \begin{bmatrix} 45.6697 & 9.1217 \\ 9.1217 & 36.1693 \end{bmatrix}$$

Since $0 < P = P^t$ and $0 < Q = Q^t$, this means that the water quality model under consideration is globally asypmtotically stable independent-of-delay. An interpretation of this result is that no matter how strong the interaction amongst the water quality constituents is, the pollution level in water remains close to the acceptable standard level.

Next, to illustrate Lemma 2 for the same system, we first compute $\lambda(A_o + E_o) = -0.7225, -2.4775$ which implies that Assumption 2 is satisfied. Then we choose $\varepsilon = 0.2$ and $\alpha = 0.1$ and use the LMI solver to determine X and τ^* that satisfy inequality (13). The feasible result is

$$X = \begin{bmatrix} 0.0751 & -0.0733 \\ -0.0733 & 0.11 \end{bmatrix}, \quad \tau^* - 1.91214$$

Since $0 < X = X^t$, this means that the time-delay system under consideration is globally asypmtotically stable for any τ satifying $0 \leq \tau \leq 1.9124$. For a given $\tau = 0.2$, the solution of LMI (15) with $\tau^* = \tau$ is given by

$$X = \begin{bmatrix} 0.3793 & -0.3954 \\ -0.3954 & 0.8844 \end{bmatrix}$$

This result implies that the pollution level in the water stream remains close to the acceptable (standard) level so long as the concentrations of the water quality constituents beyond a length $s = v \times \tau^*$ of the reach are of negligable value.

We are now in a position to deal with the concept of passivity in the context of time-delay systems.

Definition 1 The time-delay system (Σ_Δ) is called *passive* if

$$\int_0^\infty w^t(t)z(t) > \beta \quad \forall w \in L_2[0,\infty) \tag{16}$$

where β is some constant depending on the initial condition of the system. It is said to be strictly passive (SP) if it is passive and $D_o + D_o^t > 0$.

In the sequel, we shall develop conditions under which systems with time-varying parameter uncertainty and unknown state-delay like (Σ_Δ) can be guaranteed to be SP. First, we provide the following result pertaining to system (Σ_o):

THEOREM 1 *System (Σ_o) satisfying Assumption 1 is asymptotically stable with SP independent-of-delay if there exist matrices $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$ satisfying the LMI*

$$\begin{bmatrix} PA_o + A_o^tP + Q & PE_o & (C_o^t - PB_o) \\ E_o^tP & -Q & 0 \\ (C_o - B_o^tP) & 0 & -(D_o + D_o^t) \end{bmatrix} < 0 \tag{17}$$

or equivalently the matrix $(D_o + D_o^t) > 0$ and there exist matrices $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$ satisfying the ARI

$$\begin{aligned} &PA_o + A_o^tP + (C_o^t - PB_o)(D_o + D_o^t)^{-1}(C_o - B_o^tP) \\ &+ Q + PE_oQ^{-1}E_o^tP < 0 \end{aligned} \tag{18}$$

Proof See the Appendix.

In this situation, the system (Σ_o) is said to be *strongly stable with SP*. Obviously, strong stability with SP implies that system (Σ_o) is asymptotically stable with SP.

Had we modified the stability analysis to include bounds on the time-delay factor, we would then establish the following result:

THEOREM 2 *System (Σ_o) satisfying Assumption 2 is asymptotically stable with SP for any τ satisfying $0 \le \tau \le \tau^*$ if there exist matrix $0 < X = X^t \in \Re^{n\times n}$ and scalars $\varepsilon > 0, \alpha > 0, \sigma > 0$ satisfying the algebraic inequalities:*

$$(D_o + D_o^t - \tau^* \sigma^{-1} B_o^t B_o) > 0$$

and

$$\begin{aligned}(A_o + E_o)X + X(A_o + E_o)^t + \tau^* X\{\varepsilon^{-1} A_o^t A_o + \alpha^{-1} E_o^t E_o\}X \\ + \tau^*(\varepsilon + \alpha + \sigma)E_o E_o^t + (XC_o^t - B_o) \\ \times (D_o + D_o^t - \tau^* \sigma^{-1} B_o^t B_o)^{-1}(C_o X - B_o^t) < 0 \end{aligned} \tag{19}$$

or equivalently there exist matrix $0 < X = X^t \in \Re^{n \times n}$ *and scalars* $\varepsilon > 0, \alpha > 0, \sigma > 0$ *satisfying the LMI:*

$$\begin{bmatrix} (A_o + E_o)X + X(A_o + E_o)^t & \tau^* XE_o^t & \tau^* XA_o^t & (XC_o^t - B_o) & 0 \\ +\tau^*(\varepsilon + \alpha + \sigma)E_o E_o^t & & & & \\ \tau^* E_o X & -(\tau^*\alpha)I & 0 & 0 & 0 \\ \tau^* A_o X & 0 & -(\tau^*\varepsilon)I & 0 & 0 \\ (C_o X - B_o^t) & 0 & 0 & -(D_o + D_o^t) & \tau^* B_o^t \\ 0 & 0 & 0 & \tau^* B_o & -(\tau^*\sigma)I \end{bmatrix} < 0 \tag{20}$$

Proof See the Appendix.

A numerical example is in order.

4.3 Example 2

In order to illustrate Theorem 1, we consider a water-quality model of the type (10)–(11) with

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad E_o = \begin{bmatrix} 0 & 0.3 \\ -0.3 & -0.2 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, \quad C_o = [2 \quad 0], \quad D_o = 2$$

From Example 1, we know that Assumption 1 is satisfied. Using the LMI solver, it is found that the feasible solution of the LMI (17) is given by:

$$P = \begin{bmatrix} 1.2514 & 1.0697 \\ 1.0697 & 4.4010 \end{bmatrix}, \quad Q = \begin{bmatrix} 2.8590 & 0.9980 \\ 0.9980 & 1.3660 \end{bmatrix}$$

Since $0 < P = P^t$, $0 < Q = Q^t$ and $(D_o + D_o^t) > 0$, this means that the water-quality model under consideration is asypmtotically stable with SP independent-of-delay.

Next, to illustrate Theorem 2 for the same water-quality model and based on the fact that Assumption 2 is satisfied, we proceed to choose $\varepsilon = 0.2, \sigma = 0.15, \alpha = 0.1$ and use the LMI solver to determine X and τ^* that satisfy inequality (20). The result is

$$X = \begin{bmatrix} 0.2813 & -0.2062 \\ -0.2062 & 0.3308 \end{bmatrix}, \quad \tau^* = 0.3612$$

Since $0 < X = X^t$, this means that the water-quality model under consideration is asypmtotically stable with SP for any τ satisfying $0 \le \tau \le 0.3612$.

5 μ-PARAMETRIZATION

In this section, we examine the application of the stability and passivity concepts. First, motivated by the results of Theorem 1 for stability independent of delay criteria, we pose the following definition:

Definition 2 The uncertain time-delay system (Σ_Δ) is said to be strongly robustly stable with SP if there exists a matrix $0 < P = P^t \in \Re^{n \times n}$ such that for all admissible uncertainties:

$$\begin{bmatrix} PA_\Delta + A_\Delta^t P + Q & PE_o & (C_\Delta^t - PB_o) \\ E_o^t P & -Q & 0 \\ (C_\Delta - B_o^t P) & 0 & -(D_o + D_o^t) \end{bmatrix} < 0 \qquad (21)$$

Remark 1 It is readily evident from Definition 2 that the concept of strong robust stability with SP implies both the robust stability and the SP for system (Σ_Δ). Note that the robust stability with SP is an extension of quadratic stability (QS) for uncertain time-delay system to deal with the extended strict passivity problem [10]. By setting $\Delta(t) = 0$, Definition 2 reduces to (17).

Now it is easy to realize that direct application of (21) would require tremendous efforts over all admissible uncertainties. To bypass this shortcoming, we introduce the following μ-parameterized linear time-invariant system:

$$(\Sigma_\mu)\colon \dot{x}(t) = A_o x(t) + B_\mu \breve{w}(t) + E_o x(t-\tau) \tag{22}$$

$$\breve{z}(t) = C_\mu x(t) D_\mu \breve{w}(t) \tag{23}$$

where

$$B_\mu = [B_o \; 0 \; -\mu H_1] \quad C_\mu = \begin{bmatrix} C_o \\ \mu^{-1} E \\ 0 \end{bmatrix} \tag{24}$$

$$D_\mu = \begin{bmatrix} D_o & 0 & -\mu H_2 \\ 0 & 1/2I & 0 \\ 0 & 0 & 1/2I \end{bmatrix} \tag{25}$$

Remark 2 It should be stressed that the advantages of using the μ-parameterized system in the analysis of uncertain systems have been demonstrated in [9,16,20]. The main purpose is to deal with an expanded system which does not contain uncertain parameters and as it will be shown in the sequel, it provides a convenient way in establishing the technical results.

The next theorem shows that the robust SP of system (Σ_Δ) can be ascertained from the strong stability with SP of (Σ_μ).

THEOREM 3 *If there exists $\mu > 0$ such that (Σ_μ) is strongly stable with SP then system (Σ_Δ) satisfying (4)–(5) is strongly robustly stable with SP.*

Proof By Theorem 1, system (Σ_μ) is strongly stable with SP if there exist matrices $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$ such that

$$\begin{bmatrix} PA_o + A_o^t P + Q & PE_o & (C_\mu^t - PB_\mu) \\ E_o^t P & -Q & 0 \\ (C_\mu - B_\mu^t P) & 0 & -(D_\mu + D_\mu^t) \end{bmatrix} < 0 \tag{26}$$

Using (24)–(25), inequality (26) is equivalent to:

$$W = \begin{bmatrix} PA_o + A_o^tP + Q & PE_o & (C_o^t - PB_o) & \mu^{-1}E^t & \mu PH_1 \\ E_o^tP & -Q & 0 & 0 & 0 \\ (C_o - B_o^tP) & 0 & -(D_o + D_o^t) & 0 & \mu H_2 \\ \mu^{-1}E & 0 & 0 & -I & 0 \\ \mu H_1^tP & 0 & \mu H_2^t & 0 & -I \end{bmatrix} < 0 \tag{27}$$

By Fact 1, inequality (27) holds if and only if

$$\Omega_1 + \Omega_3^t\Omega_2^{-1}\Omega_3 < 0 \tag{28}$$

with

$$\Omega_1 = \begin{bmatrix} PA_o + A_o^tP + Q & PE_o & (C_o^t - PB_o) \\ E_o^tP & -Q & 0 \\ (C_o - B_o^tP) & 0 & -(D_o + D_o^t) \end{bmatrix};$$

$$\Omega_3^t = \begin{bmatrix} \mu^{-1}E^t & \mu PH_1 \\ 0 & 0 \\ 0 & \mu H_2 \end{bmatrix} \tag{29}$$

$$\Omega_2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \tag{30}$$

It follows from a well-known result in [9] that inequality (28) along with (29)–(30) holds if and only if

$$\Omega_1 + \begin{bmatrix} PH_1 \\ 0 \\ H_2 \end{bmatrix}\Delta(t)[E\ 0\ 0] + \begin{bmatrix} E^t \\ 0 \\ 0 \end{bmatrix}\Delta^t(t)[H_1^tP\ 0\ H_2^t] < 0 \tag{31}$$

$$\forall \Delta : \Delta^t\Delta \le I \tag{32}$$

Rearranging (31) using (6) and (29)–(30), it produces

$$\begin{bmatrix} PA_\Delta + A_\Delta^t P + Q & PE_o & (C_\Delta^t - PB_o) \\ E_o^t P & -Q & 0 \\ (C_\Delta - B_o^t P) & 0 & -(D_o + D_o^t) \end{bmatrix} < 0 \tag{33}$$

which in view of Definition 2 implies that system (Σ_Δ) is strongly stable with SP.

Remark 3 One way to evaluate the robust stability of the uncertain time-delay system (Σ_Δ) with SP is to rewrite inequality (27) with $\rho = \mu^{-2}$ in the form:

$$W_t = \begin{bmatrix} G(P) & L(P,\rho) \\ L^t(P,\rho) & U \end{bmatrix} < 0 \tag{34}$$

where

$$G(P) = \begin{bmatrix} PA_o + A_o^t P + Q & PE_o & C_o^t - PB_o \\ E_o^t P & -Q & 0 \\ (C_o - PB_o^t) & 0 & -(D_o + D_o^t) \end{bmatrix} < 0 \tag{35}$$

$$L(P,\rho) = \begin{bmatrix} \rho E^t & PH_1 \\ 0 & 0 \\ 0 & H_2 \end{bmatrix}, \quad U = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \tag{36}$$

Obviously, (34)–(36) is linear in P and ρ which can be solved by employing the LMI Toolbox [11].

5.1 Example 3

In order to illustrate Theorem 3, we consider an uncertain water-quality system of the type (4)–(5) with

$$A_o = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad E_o = \begin{bmatrix} 0 & 0.3 \\ -0.3 & -0.2 \end{bmatrix}$$

$$B_o = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, \quad C_o = [2 \quad 0], \quad D_o = 2$$

$$H_1 = \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}, \quad E = [0.1 \quad 0.2], \quad H_2 = 0.1$$

From Example 1, we know that Assumption 1 is satisfied. Using the LMI solver, it is found that feasible solutions of the LMI (34) are given by:

$$\rho = 0.4, \quad \begin{bmatrix} 0.8491 & 0.6459 \\ 0.6459 & 2.6943 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.8243 & 0.7333 \\ 0.7333 & 0.9264 \end{bmatrix}$$

$$\rho = 0.9, \quad \begin{bmatrix} 0.8516 & 0.6501 \\ 0.6501 & 2.7012 \end{bmatrix}, \quad Q = \begin{bmatrix} 1.8263 & 0.7342 \\ 0.7342 & 0.9267 \end{bmatrix} \tag{37}$$

Since $0 < P = P^t$, $0 < Q = Q^t$ and $(D_o + D_o^t) > 0$, this means that the uncertain water-quality system under consideration is strongly robustly stable with SP independent-of-delay.

Had we adopted the delay-dependent stability criteria, we would then adopt the following definition:

Definition 3 The uncertain time-delay system (Σ_Δ) is said to be strongly robustly stable with SP for any τ satisfying $0 \le \tau \le \tau^*$ if there exist a matrix $0 < X = X^t \in \Re^{n \times n}$ and scalars $\varepsilon > 0$, $\alpha > 0$, $\sigma > 0$ satisfying the LMI

$$\begin{bmatrix} \begin{matrix} (A_\Delta + E_o)X \\ +X(A_\Delta + E_o)^t \\ +\tau^*(\varepsilon + \alpha + \alpha)E_o E_o^t \end{matrix} & \tau^* X E_o^t & \tau^* X A_\Delta^t & (XC_\Delta^t - B_o) & 0 \\ \tau^* E_o X & -(\tau^*\alpha)I & 0 & 0 & 0 \\ \tau^* A_\Delta X & 0 & -(\tau^*\varepsilon)I & 0 & 0 \\ (C_\Delta X - B_o^t) & 0 & 0 & -(D_o + D_o^t) & -\tau^* B_o^t \\ 0 & 0 & 0 & -\tau^* B_o & (\tau^*\sigma)I \end{bmatrix} < 0 \tag{38}$$

It should be observed that by setting $\Delta(t) \equiv 0$, Definition 3 reduces to (16).

The next result is a delay-dependent counterpart of Theorem 3.

THEOREM 4 *System* (Σ_Δ) *satisfying (4)–(5) is said to be strongly robustly stable with SP for any* τ *satisfying* $0 \leq \tau \leq \tau^*$ *if there exists a* $\mu > 0$ *satisfying the inequality:*

$$\begin{bmatrix} \Xi_1 & \Xi_3^t \\ \Xi_3 & U \end{bmatrix} < 0 \tag{39}$$

where

$$\Xi_1 = \begin{bmatrix} \begin{array}{c} (A_o + E_o)X \\ +X(A_o + E_o)^t \\ +\tau^*(\varepsilon + \alpha + \sigma)E_o E_o^t \end{array} & \tau^* X E_o^t & \tau^* X A_\Delta^t & (XC_o^t - B_o) & 0 \\ \tau^* E_o X & -(\tau^*\alpha)I & 0 & 0 & 0 \\ \tau^* A_o X & 0 & -(\tau^*\varepsilon)I & 0 & 0 \\ (C_o X - B_o^t) & 0 & 0 & -(D_o + D_o^t) & -\tau^* B_o^t \\ 0 & 0 & 0 & -\tau^* B_o & (\tau^*\sigma)I \end{bmatrix}$$

$$\Xi_3^t = \begin{bmatrix} H_1 & XE^t \\ 0 & 0 \\ \tau^* H_1 & 0 \\ H_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \tag{40}$$

Proof Note that (40) together with Fact 2 implies

$$\Xi_1 + \begin{bmatrix} H_1 \\ 0 \\ \tau^* H_1 \\ H_2 \\ 0 \end{bmatrix} \Delta(t)[EX\ 0\ 0\ 0\ 0] + \begin{bmatrix} XE^t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta^t(t)[H_1^t\ 0\ \tau^* H_1\ H_2^t\ 0] < 0$$

$$\forall \Delta : \Delta^t \Delta \leq I \tag{41}$$

That is, (38) holds. By Definition 3, the system (Σ_Δ) is strongly robustly stable with SP. Note that Theorem 4 is basically an LMI feasibility result.

6 OBSERVER-BASED CONTROL SYNTHESIS

The analysis of robust stability with SP can be naturally extended to the corresponding synthesis problem. That is, we are concerned with the design of a feedback controller that not only internally stabilizes the uncertain time-delay system but also achieves SP for all admissible uncertainties and unknown delays. A controller which achieves the property of robust stability with SP is termed as a robust SP controller. To this end, we consider the class of uncertain systems of the form:

$$(\Sigma_{co})\ \dot{x}(t) = A_\Delta x(t) + B_o w(t) + B_{1\Delta} u(t) + E_o x(t-\tau) \tag{42}$$

$$z(t) = C_\Delta x(t) + D_o w(t) + D_{12\Delta} u(t) \tag{43}$$

$$y(t) = C_{1\Delta} x(t) + D_{21} w(t) + D_{22\Delta} u(t) \tag{44}$$

where $y \in \Re^n$ is the measured output and $u \in \Re^n$ is the control input. The uncertain matrices are given by:

$$\begin{bmatrix} A_\Delta & B_{1\Delta} \\ C_\Delta & D_{12\Delta} \\ C_{1\Delta} & D_{22\Delta} \end{bmatrix} = \begin{bmatrix} A_o & B_1 \\ C_o & D_{12} \\ C_1 & D_{22} \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \Delta(t)[\ E_1\ E_3\ ] \tag{45}$$
$$\Delta^t(t)\Delta(t) \le I \quad \forall t$$

In the sequel, we focus attention on the controller synthesis for system (Σ_{co}) by using an observer-based controller of the form

$$(\Sigma_{ob}) :\ \dot{\eta}(t) = G_o \eta(t) + L_o[y(t) - C_1 \eta(t)] \tag{46}$$

$$u(t) = K_o \eta(t) \tag{47}$$

where (G_o, L_o, K_o) are constant matrices to be selected. Define the augmented state-vector by:

$$\xi(t) = \begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} \tag{48}$$

Applying the observer-based controller (46)–(47) to system (Σ_{co}) and using (42), we obtain the closed-loop system:

$$(\Sigma_{cc}) : \ \dot{\xi}(t) = \overset{\frown}{A}_\Delta \xi(t) + \tilde{B} w(t) + \overset{\frown}{E}\, \xi(t-\tau) \tag{49}$$

$$z(t) = \overset{\frown}{C}_\Delta \xi(t) + D_o w(t) \tag{50}$$

where

$$\begin{aligned} \overset{\frown}{A}_\Delta &= \begin{bmatrix} A_\Delta & B_{1\Delta} K_o \\ L_o C_{1\Delta} & G_o - L_o C_1 + L_o D_{22\Delta} K_o \end{bmatrix} \\ &= \begin{bmatrix} A_o & B_1 K_o \\ L_o C_1 & G_o - L_o C_1 + L_o D_{22} K_o \end{bmatrix} + \tilde{H}\Delta(t)\tilde{E} \\ &= \hat{A} + \tilde{H}\Delta(t)\tilde{E} \end{aligned} \tag{51}$$

$$\tilde{B} = \begin{bmatrix} B_o \\ L_o D_{21} \end{bmatrix} \tag{52}$$

$$\begin{aligned} \overset{\frown}{C}_\Delta &= [C_\Delta \ \ D_{12\Delta} K_o] \\ &= \tilde{C} + H_2 \Delta(t) \tilde{E}, \quad \overset{\frown}{E} = \begin{bmatrix} E_o & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{53}$$

and

$$\tilde{H} = \begin{bmatrix} H_1 \\ L_o H_3 \end{bmatrix}, \quad \tilde{E} = [E_1 \ \ E_3 K_o], \quad \tilde{C} = [C_o \ \ D_{12} K_o] \tag{54}$$

On the other hand, we introduce the following μ-parameterized linear time-invariant system:

$$(\Sigma_{\mu\mu})\ \dot{x}(t) = A_o x(t) + B_\mu \breve{w}(t) + B_1 u(t) + E_o x(t-\tau) \tag{55}$$

$$\breve{z}(t) = C_\mu x(t) + D_\mu \breve{w}(t) + D_{1\mu} u(t) \tag{56}$$

$$y(t) = C_1 x(t) + D_{2\mu} \breve{w}(t) + D_{22} u(t) \tag{57}$$

where

$$D_{1\mu} = \begin{bmatrix} D_{12} \\ \mu^{-1} E_3 \\ 0 \end{bmatrix}, \quad D_{2\mu} = [D_{21}\ 0\ -\mu H_3] \tag{58}$$

and B_μ, C_μ, D_μ, are given by (24)–(25). Now by combining systems $(\Sigma_{\mu\mu})$ and (Σ_{ob}), we obtain the closed-loop μ-parameterized system $(\Sigma_{c\mu})$:

$$(\Sigma_{c\mu}):\ \dot{\xi}(t) = \widehat{A}\,\xi(t) + \hat{B}\breve{w}(t) + \widehat{E}\,\xi(t-\tau) \tag{59}$$

$$z(t) = \widehat{C}\,\xi(t) + D_\mu \breve{w}(t) \tag{60}$$

where

$$\hat{B} = \begin{bmatrix} B_\mu \\ L_o D_{2\mu} \end{bmatrix} \tag{61}$$

$$\hat{C} = [C_\mu\ D_{1\mu} K_o] \tag{62}$$

The next theorem provides an interconnection between the observer-based passive real control of system (Σ_{cc}) and the passive real control of system $(\Sigma_{c\mu})$.

THEOREM 5 *If for some $\mu > 0$ an observer-based controller of the form (46)–(47) achieves strong stability with SP for system $(\Sigma_{c\mu})$, then this observer-based controller achieves strong robust stability with SP for system (Σ_Δ) for all admissible uncertainties satisfying (6)–(7).*

Proof By Theorem 1, system $(\Sigma_{c\mu})$ is strongly stable with SP if there exist matrices $0 < X = X^t$ and $0 < \hat{Q} = \hat{Q}^t$ such that

$$\begin{bmatrix} X\hat{A} + \hat{A}^t X + \hat{Q} & X\widehat{E} & (\hat{C}^t - X\hat{B}) \\ \widehat{E}^t X & -\hat{Q} & 0 \\ (\hat{C} - \hat{B}^t X) & 0 & -(D_\mu + D_\mu^t) \end{bmatrix} < 0 \tag{63}$$

Expansion of (63) using (24)–(25) and (53)–(56) yields:

$$\begin{bmatrix} X\hat{A} + \hat{A}^tX + \hat{Q} & X\widehat{E} & (\tilde{C}^t - X\tilde{B}) & \mu^{-1}\tilde{E}^t & \mu X\tilde{H} \\ \widehat{E}^tX & -\hat{Q} & 0 & 0 & 0 \\ (\tilde{C} - \tilde{B}^tX) & 0 & -(D_o + D_o^t) & 0 & \mu H_2 \\ \mu^{-1}\tilde{E} & 0 & 0 & -I & 0 \\ \mu\tilde{H}^tX & 0 & \mu H_2^t & 0 & -I \end{bmatrix} < 0 \quad (64)$$

Applying Fact 1, inequality (64) holds if and only if

$$\Omega_1 + \Omega_3^t\Omega_2^{-1}\Omega_3 < 0 \quad (65)$$

with

$$\Omega_1 = \begin{bmatrix} X\hat{A} + \hat{A}^tX + \hat{Q} & X\widehat{E} & (\tilde{C}^t - X\tilde{B}) \\ \widehat{E}^tX & -\hat{Q} & 0 \\ (\tilde{C} - \tilde{B}^tX) & 0 & -(D_o + D_o^t) \end{bmatrix} \quad (66)$$

$$\Omega_3^t = \begin{bmatrix} \mu^{-1}\tilde{E}^t & \mu X\tilde{H} \\ 0 & 0 \\ 0 & \mu H_2 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix} \quad (67)$$

Then by Fact 2 inequality (65) along with (66)–(67) holds if and only if

$$\Omega_1 + \begin{bmatrix} X\tilde{H} \\ 0 \\ H_2 \end{bmatrix}\Delta(t)[\tilde{E} \;\; 0 \;\; 0] + \begin{bmatrix} \tilde{E}^t \\ 0 \\ 0 \end{bmatrix}\Delta^t(t)[\tilde{H}^t \;\; X \;\; 0 \;\; H_2^t] < 0 \quad \forall\Delta(t) \leq I \quad (68)$$

or equivalently

$$\begin{bmatrix} X\hat{A} + \hat{A}^t X + \hat{Q} + & X\widehat{E} & (\tilde{C}^t - X\tilde{B}) - \\ X\tilde{H}\Delta(t)\tilde{E} + \tilde{E}^t\Delta^t(t)\tilde{H}^t X & & + \tilde{E}^T\Delta^t(t)H_2^t \\ \widehat{E}^t X & -\hat{Q} & 0 \\ (\tilde{C} - \tilde{B}^t X) + & 0 & -(D_o + D_o^t) \\ H_2\Delta(t)\tilde{E} & & \end{bmatrix} < 0 \tag{69}$$

for all admissible uncertainties. The substitution of (53)–(56) into inequality (69) produces:

$$\begin{bmatrix} X\widehat{A}_\Delta + \widehat{A}^t{}_\Delta X + \hat{Q} & X\widehat{E} & (\widehat{C}^t{}_\Delta - X\tilde{B}) \\ \widehat{E}^t X & -\hat{Q} & 0 \\ (\widehat{C}_\Delta - \tilde{B}^t X) & 0 & -(D_o + D_o^t) \end{bmatrix} < 0 \tag{70}$$

which in view of Theorem 3 implies that system (Σ_{cc}) is strongly robustly stable with SP.

Remark 4 In general, the gain matrices of the observer-based controller (46)–(47) can be determined by appropriately modifying the results of [8, Theorem 4.1] to include the additional quadratic terms due to the state-delay.

7 CONCLUSIONS

This paper has provided techniques of passivity analysis and control synthesis of a class of linear dynamical systems with norm-bounded uncertainties and state-delay. For the analysis problem, we have derived a sufficient condition for which the uncertain time-delay system is robustly stable and strictly passive for all admissible uncertainties. This condition is expressed in terms of a linear matrix inequality. Both the cases of delay-independent and delay-dependent have been considered. For the synthesis problem, an observer-based design method has been proposed which guarantees that the closed-loop uncertain time-delay system is stable and strictly passive for all admissible uncertainties.

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8 APPENDIX

Proof of Lemma 1

(1) Introduce a Lyapunov-Krasovskii functional $V_1(x)$ of the form:

$$V_1(x) = x^t(t)Px(t) + \int_{t-\tau}^{t} x^t(\theta)Qx(\theta)\,d\theta \tag{71}$$

where $0 < P = P^t \in \Re^{n\times n}$ and $0 < Q = Q^t \in \Re^{n\times n}$ are weighting matrices. By differentiating $V_1(x)$ along the solutions of (1) and arranging terms, we get:

$$\dot{V}_1(x) = [x^t(t)\ x^t(t-\tau)]\,\Omega_f\begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{72}$$

where

$$\Omega_f = \begin{bmatrix} PA_o + A_o^tP + Q & PE_o \\ E_o^tP & -Q \end{bmatrix} \tag{73}$$

If $\dot{V}_1(x) < 0$, when $x \neq 0$ then $x(t) \to 0$ as $t \to \infty$ and the time-delay system (Σ_f) is globally asymptotically stable independent of delay. From (68)–(69), this stability condition is guaranteed if inequality (9) holds.

(2) By the Schur complement formula, (10) is equivalent to (9).

Proof of Lemma 2

(1) Introduce a Lyapunov-Krasovskii functional $V_2(x)$ of the form:

$$V_2(x) = x^t(t)Px(t) + \int_{t-\tau}^{t}\left\{\int_{t+\theta}^{t} r_1[x^t(s)A_o^tA_ox(s)]ds \right.$$
$$\left. + \int_{t-\tau+\theta}^{t} r_2x[s^t(s)E_o^tE_ox(s)]ds\right\}d\theta \tag{74}$$

where $0 < P = P^t \in \Re^{n\times n}$ and $r_1 > 0$, $r_2 > 0$ are weighting factors. First from (1) we have

$$x(t-\tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t+\theta)d\theta$$

$$= x(t) - \int_{-\tau}^{0} A_o x(t+\theta)d\theta - \int_{-\tau}^{0} E_o x(t-\tau+\theta)d\theta \quad (75)$$

Substituting (71) back into (1) we get:

$$\dot{x}(t) = [A_o + E_o]x(t) - E_o\left\{\int_{-\tau}^{0} A_o x(t+\theta)d\theta + \int_{-\tau}^{0} E_o x(t-\tau+\theta)d\theta\right\} \quad (76)$$

Now by differentiating $V_2(x)$ along the solutions of (72) and arranging terms, we obtain:

$$\begin{aligned}\dot{V}_2(x) = & x^t(t)[P(A_o + E_o) + (A_o + E_o)^t P]x(t) \\ & - 2x^t(t)PE_o \int_{-\tau}^{0} A_o x(t+\theta)d\theta \\ & - 2x^t(t)PE_o \int_{-\tau}^{0} E_o x(t-\tau+\theta)d\theta + \tau r_1 x^t(t)A_o^t A_o x(t) \\ & + \tau r_2 x^t(t)E_o^t E_o x(t) - \int_{-\tau}^{0} r_1[x^t(t+\theta)A_o^t A_o x(t+\theta)]d\theta \\ & - \int_{-\tau}^{0} r_2[x^t(t-\tau+\theta)E_o^t E_o x(t-\tau+\theta)]d\theta \end{aligned} \quad (77)$$

Note that

$$
\begin{aligned}
&- 2x^t(t)PE_o \int_{-\tau}^{0} A_o x(t+\theta)d\theta \le r_1^{-1} \int_{-\tau}^{0} [x^t(t)PE_oE_o^tPx(t)]d\theta \\
&+ r_1 \int_{-\tau}^{0} [x^t(t+\theta)A_o^tA_ox(t+\theta)]d\theta \\
&\quad = \tau r_1^{-1} x^t(t)PE_oE_o^tPx(t) + r_1 \int_{-\tau}^{0} [x^t(t+\theta)A_o^tA_ox(t+\theta)]d\theta \qquad (78)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&- 2x^t(t)PE_o \int_{-\tau}^{0} E_o x(t-\tau+\theta)d\theta \\
&\le r_2^{-1} \int_{-\tau}^{0} [x^t(t)PE_oE_o^tPx(t)]d\theta \\
&\quad + r_2 \int_{-\tau}^{0} [x^t(t-\tau+\theta)E_o^tE_ox(t-\tau+\theta)]d\theta \\
&= \tau r_2^{-1} x^t(t)PE_oE_o^tPx(t) \\
&\quad + r_2 \int_{-\tau}^{0} [x^t(t-\tau+\theta)E_o^tE_ox(t-\tau+\theta)]d\theta \qquad (79)
\end{aligned}
$$

Hence, it follows from (73) that

$$
\begin{aligned}
\dot{V}_2(x) = x^t(t)[&P(A_o+E_o) + (A_o+E_o)^tP + \tau r_1 A_o^tA_o + \tau r_2 E_o^tE_o \\
&+ \tau r_1^{-1}PE_oE_o^tP + \tau r_2^{-1}PE_oE_o^tP]x(t) \qquad (80)
\end{aligned}
$$

If $\dot{V}_2(x) < 0$ when $x \neq 0$, then $x(t) \to 0$ as $t \to \infty$ and the time-delay system (Σ_f) is globally asymptotically stable. By defining $r_1 = \varepsilon^{-1}$ and $r_2 = \alpha^{-1}$, then it follows from (76) for any $\tau \in [0, \tau^*]$ that the stability condition is satisfied if

$$P(A_o + E_o) + (A_o + E_o)^t P + \tau^*(\varepsilon + \alpha) P E_o E_o^t P \\ + \tau^* \varepsilon^{-1} A_o^t A_o + \tau^* \alpha^{-1} E_o^t E_o < 0 \tag{81}$$

Premultiplying (77) by P^{-1}, postmultiplying the result by P^{-1} and letting $X = P^{-1}$ one can arrange the result into the block form (11) as desired.

(2) By the Schur complement formula, (11) is equivalent to (10).

Proof of Theorem 1

(1) By evaluatinging $V_1(x)$ along the solutions of (1)–(2) with some manipulations, we obtain the associated Hamilatonian $H(x, t)$:

$$\begin{aligned} H(x,t) &= -\dot{V}_1(x) + 2y^t(t)u(t) \\ &= -x^t(t)[PA_o + A_o^t P + Q]x(t) + u^t(t)(D_o + D_o^t)u(t) \\ &\quad + x^t(t-\tau)Qx(t-\tau) - x^t(t-\tau)E_o^t P x(t) - x^t(t) P E_o x(t-\tau) \\ &\quad + u^t(t)(C_o - B_o^t P)x(t) + x^t(t)(C_o^t - PB_o)u(t) \end{aligned} \tag{82}$$

In terms of the augmented state vector $Z(t) = [x^t(t)\ x^t(t-\tau)\ u^t(t)]^t$, we express $H(x, t)$ as

$$H(x,t) = -Z^t(t)\ \Omega(P)\ z(t) \tag{83}$$

where

$$\Omega(P) = \begin{bmatrix} PA_o + A_o^t P + Q & PE_o & (C_o^t - PB_o) \\ E_o^t P & -Q & 0 \\ (C_o - B_o^t P) & 0 & -(D_o + D_o^t) \end{bmatrix} \tag{84}$$

If $\Omega(P) < 0$, then $-\dot{V}_1(x) + 2y^t(t)u(t) > 0$, and from which it follows that

$$\int_{t_o}^{t_1} [y^t(t)u(t)]dt > 1/2[V(x(t_1)) - V(x(t_o))] \tag{85}$$

Since $V_1(x) > 0$ for $x \neq 0$ and $V(x) = 0$ for $x = 0$, it follows that as $t_1 \to \infty$ that system (Σ_d) is extended strictly passive. This proves inequality (15).

Proof of Theorem 2 Introduce the following Lyapunov-Krasvoskii functional $V_3(x)$:

$$V_3(x) = x^t(t)Px(t) + \int_{-\tau}^{t} \left\{ \int_{t+\theta}^{t} r_1[x^t(s)A_o^tA_ox(s)]ds + \int_{t-\tau+\theta}^{t} r_2[x^t(s)E_o^tE_ox(s)]ds + \int_{t+\theta}^{t} r_3[u^t(s)B_o^tB_ou(s)]ds \right\} d\theta \qquad (86)$$

where $0 < P = P^t \in \Re^{n\times n}$ and $r_1 > 0$, $r_2 > 0$, $r_3 > 0$ are weighting factors. Now from (1) we get

$$\begin{aligned} x(t-\tau) &= x(t) - \int_{-\tau}^{0} \dot{x}(t+\theta)d\theta \\ &= x(t) - \int_{-\tau}^{0} A_ox(t+\theta)d\theta - \int_{-\tau}^{0} E_ox(t-\tau+\theta)d\theta \\ &\quad - \int_{-\tau}^{0} B_ou(t+\theta)d\theta \end{aligned} \qquad (87)$$

Hence, the state dynamics (1) becomes:

$$\begin{aligned} \dot{x}(t) &= [A_o + E_o]x(t) + B_ou(t) \\ &\quad - E_o\left[\int_{-\tau}^{0} A_ox(t+\theta)d\theta + \int_{-\theta}^{0} E_ox(t-\tau+\theta)d\theta \right. \\ &\quad \left. + \int_{-\tau}^{0} B_ou(t+\theta)d\theta\right] \end{aligned} \qquad (88)$$

By differentiating $V_3(x)$ along the solutions of (73) and arranging terms, we obtain:

$$\dot{V}_3(x) = x^t(t)[P(A_o + E_o) + (A_o + E_o)^t P]x(t) + u^t(t)B_o^t Px(t)$$
$$+ x^t(t)PB_o u(t) - 2x^t(t)PE_o \int_{-\tau}^{0} A_o x(t+\theta)d\theta$$
$$- 2x^t(t)PE_o \int_{-\tau}^{0} E_o x(t+\theta)d\theta - 2x^t(t)PE_o \int_{-\tau}^{0} B_o u(t+\theta)d\theta$$
$$+ \tau r_1 x^t(t)A_o^t A_o x(t) + \tau r_2 x^t(t)E_o^t E_o x(t) + \tau r_3 u^t(t)B_o^t B_o u(t)$$
$$- \int_{-\tau}^{0} r_1[x^t(t+\theta)A_o^t A_o x(t+\theta)]d\theta$$
$$- \int_{-\tau}^{0} r_2[x^t(t-\tau+\theta)E_o^t E_o x(t-\tau+\theta)]d\theta$$
$$- \int_{-\tau}^{0} r_3[u^t(t+\theta)B_o^t B_o u(t+\theta)]d\theta \qquad (89)$$

By considering

$$- 2x^t(t)PE_o \int_{-\tau}^{0} B_o u(t+\theta)d\theta \leq \tau r_3^{-1} x^t(t)PE_o E_o^t Px(t)$$
$$+ r_3 \int_{-\tau}^{0} [u^t(t+\theta)B_o^t B_o u(t+\theta)]xd\theta \qquad (90)$$

and substituting inequality (74) into (74), there holds

$$\dot{V}_3(x) = x^t(t)[P(A_o + E_o) + (A_o + E_o)^t P + \tau r_1 A_o^t A_o + \tau r_2 E_o^t E_o$$
$$+ \tau r_1^{-1} PE_o E_o^t P + \tau r_2^{-1} PE_o E_o^t P + \tau r_3^{-1} PE_o E_o^t P]x(t)$$
$$+ u^t(t)B_o^t Px(t) + x^t(t)PB_o u(t) + \tau r_3 u^t(t)B_o^t B_o u(t) \qquad (91)$$

Using (76), the associated Hamiltonian $H(x, t)$ can be written as:

$$
\begin{aligned}
H(x,t) = -x^t(t)[&P(A_o+E_o) + (A_o+E_o)^t P + \tau r_1 A_o^t A_o \\
&+ \tau r_2 E_o^t E_o + \tau r_1^{-1} P E_o E_o^t P \\
&+ \tau r_2^{-1} P E_o E_o^t P + \tau r_3^{-1} P E_o E_o^t P]x(t) - \tau r_3 u^t(t) B_o^t B_o u(t) \\
&+ u^t(t)(D_o + D_o^t)u(t) + x^t(t)[C_o^t - PB_o]u(t) \\
&+ u^t(t)[C_o - B_o^t P]x(t)
\end{aligned}
\tag{92}
$$

A little algebra puts $H(x,t)$ in the compact form:

$$
H(t) = -Y^t(t)\ \Pi(P)\ Y(t) \tag{93}
$$

where

$$
Y(t) = [x^t(t)\ u^t(t)]^t \tag{94}
$$

$$
\Pi(P) = \begin{bmatrix} S(P) & (C_o^t - PB_o) \\ (C_o - B_o^t P) & -(D_o + D_o^t - \tau r_3 B_o^t B_o) \end{bmatrix} < 0 \tag{95}
$$

$$
\begin{aligned}
S(P) = P(A_o+E_o) + (A_o+E_o)^t P + \tau(r_1 A_o^t A_o + r_2 E_o^t E_o) \\
+ \tau(r_1^{-1} + r_2^{-1} + r_3^{-1}) P E_o E_o^t P
\end{aligned}
\tag{96}
$$

If $\Pi(P) < 0$, then $-\dot{V}_3 + 2y^t(t)u(t) > 0$ and from which it follows that

$$
\int_{t_o}^{t_1} [y^t(t)\ u(t)]dt > 1/2[V(x(t_1)) - V(x(t_o))] \tag{97}
$$

Since $V(x) > 0$ for $x \neq 0$ and $V(x) = 0$ for $x = 0$, it follows that as $t_1 \rightarrow \infty$ that system (Σ_d) is extended strictly passive. From the

Schur complement formula, it is easy to know that (80)–(81) is equivalent to:

$$\begin{aligned}&P(A_o + E_o) + (A_o + E_o)^t P + \tau(r_1 A_o^t A_o + r_2 E_o^t E_o)\\&\quad + \tau(r_1^{-1} + r_2^{-1} + r_3^{-1}) P E_o E_o^t P\\&\quad + (C_o^t - PB_o)(D + D^t - \tau r_3 B_o^t B_o)^{-1}(C_o - B_o^t P) < 0 \qquad (98)\end{aligned}$$

Setting $r_1 = \varepsilon^{-1}$, $r_2 = \alpha^{-1}$, $r_3 = \sigma^{-1}$, premultiplying (83) by P^{-1}, postmultiplying the result by P^{-1} and letting $X = P^{-1}$, it shows that (16) implies (83) for any $0 \leq \tau \leq \tau^*$.